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Wigner particle theory and local quantum physics*

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Abstract

Wigner's theory of positive energy representations of the Poincaré group has often been used to give additional justifications for the Lagrangian quantization approach to QFT. Here we show that by extension with a modular localization structure it can directly lead to the net of local algebras without the use of any point-like field coordinatization. The same modular methods reveal that among the irreducible representations there are two exotic types ($d = 1 + 2$ massive anyons and $d = 1 + 3$ zero mass helicity towers) whose localization is string-like; in fact, their conversion into operator algebras leads to free string field theory. We also report on two attempts to extend the underlying spirit of the intrinsic (nonquantization) Wigner approach to the realm of interacting theories. Both aim at unravelling the structure of (Rindler) wedge-localized algebras and show, for the first time, the constructive power of the algebraic approach which, although conceived by Rudolph Haag more than 40 years ago, has primarily contributed to the structural understanding of QFT.

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1. The setting of the problem

The algebraic framework of local quantum physics shares the same physical principles with the standard textbook approach to QFT but differs in concepts and tools used for their implementation. Whereas the standard approach is based on 'field coordinatizations' in terms of point-like fields (without which the canonical or functional integral quantization is hardly conceivable), the algebraic framework permits the formulation of local quantum physics directly in terms of a net of local operator algebras, i.e. without the intervention of the rather singular point-like field coordinates whose indiscriminate use is the potential source of

* Dedicated to Rudolf Haag on the occasion of his 80th birthday.

ultraviolet divergencies. Among the many advantages is the fact that the somewhat artistic[3] standard scheme is replaced by a conceptually better balanced setting.

The advantages of such an approach [1–3] were in the eyes of many particle physicist offset by its constructive weaknesses of which even its protagonists (who used it mainly for structural investigations such as TCP, spin and statistics and similar) were well aware [3]. In particular, even those formulations of renormalized perturbation theory which were closest in spirit to the algebraic approach such as the causal perturbation theory and its recent refinements [4] use a coordinatization of algebras in terms of fields at some state. The underlying 'Bogoliubov-axiomatic' [5] in terms of an off-shell generating 'S-matrix' $S(g)$ suffers apparently from the same ultraviolet limitations as any other point-like field formulation.

However, there are signs of change which are not only a consequence of the lack of success of many popular attempts in post standard model particle theory, rather it is also becoming slowly but steadily clear that the times of constructive nonperturbative weakness of the algebraic approach (AQFT) are passing and the significant conceptual investments are beginning to bear fruits for the actual construction of models.

The constructive aspects of these gains are presently most clearly visible in situations in which there is no real (on-shell) particle creation but for which, different from free field theories, the vacuum polarization structure remains very rich [6]. It is not possible in these models to locally generate one-particle states from the vacuum without accompanying vacuum polarization clouds. Besides the well-known $d = 1 + 1$ factorizing models, this includes the QFTs associated with exceptional Wigner representations, i.e. $d = 1 + 2$ 'anyonic' spin. In the latter case, the absence of compact localization renders the theories more noncommutative and in turn less accessible to Lagrangian quantization methods. The main content of this paper deals with constructive aspects of such models.

The historical roots of the algebraic approach date back to the 1939 famous Wigner paper [7] whose aim was to obtain an intrinsic conceptual understanding of particles, avoiding the ambiguous wave equation method and the closely related Lagrangian quantization so that a physical equivalence of different Lagrangian descriptions could be easily recognized. In fact, it was precisely because of this fundamental intrinsic appeal and the unicity of Wigner's approach that some authors felt compelled to present this theory as a kind of additional partial justification for the Lagrangian (canonical or functional) quantization [8].

Since the late 1950s there has been a dream about a kind of royal path into nonperturbative particle physics which starts from Wigner's representation theoretic particle setting and introduces interactions in a maximally intrinsic and invariant way. This was thought to be accomplished by using concepts which avoid performing computations in terms of the standard singular field coordinatization and which instead lean on the unitary and crossing symmetric scattering operator and the associated spaces of form factors. It is well known that this dream in its original form (a unique theory of almost everything: a TOE without gravitation based on an S-matrix bootstrap doctrine) failed, and that some of the old ideas were reprocessed and entered string theory via Veneziano's dual model. In the following, we will show that certain aspects of this old folklore (which certainly do not include that of a TOE), if enriched with new concepts, can have successful applications for the above-mentioned class of models.

According to Wigner, particles should be described by irreducible positive energy representation of the Poincaré group. In fact, they are the indecomposable building blocks of those multi-localized asymptotically stable objects in terms of which each state can be

[3] The postulated canonical or functional representation requirement is known to disappear in the course of the renormalization calculations and the physical (renormalized) result only satisfies the more general causality/locality properties.

interpreted and measured in counter(anti)coincidence arrangements in the large time limit. This raises the question what kind of localization properties particles should be expected to have, and which positive energy representations permit what kind of localization.

There are two localization concepts in particle physics. One is the 'Born-localization' taken over from Schrödinger theory which is based on probabilities and associated projectors projecting onto compactly supported subspaces of spatially localized wavefunctions at a fixed time; in the relativistic context, this quantum mechanical localization also bears the name 'Newton–Wigner' localization [1]). The incompatibility of this localization with relativistic covariance and Einstein causality was already noted and analysed by its protagonists [9]. Covariance and causality are however satisfied in the asymptotic region[4] (through the use of the asymptotic behaviour of wavefunctions) and therefore the relativistic covariance and cluster separability of the Moeller operators and the S-matrix are not affected by the use of this less-than-perfect quantum mechanical localization. On the other hand, there exists a fully relativistic covariant localization which is intimately related to the characteristic causality and vacuum polarization properties of QFT; in the standard formulation of QFT, it is the localization which is encoded in the position of the dense subspace obtained by applying smeared fields (with a fixed test function support) to the vacuum. Since in the field-free formulation of local quantum physics this localization turns out to be inexorably linked to the Tomita–Takesaki modular theory of operator algebras [1], it will be shortly referred to as 'modular localization'. Its physical content is less obvious and its consequences are less intuitive, and therefore we will take some care in its presentation.

In fact, the remaining part of this introductory section is used to contrast the Newton–Wigner localization with the modular localization. This facilitates the understanding of both concepts.

The use of Wigner's group theory based particle concept for the formulation of what has been called[5] 'direct interactions' in relativistic multiparticle systems can be nicely illustrated by briefly recalling the arguments which led to this relativistic form of macrocausal quantum mechanics. Bakamjian and Thomas [10] observed as far back as 1953 that it is possible to introduce an interaction into the tensor product space describing two Wigner particles by keeping the additive form of the total momentum $\vec{P}$, its canonical conjugate $\vec{X}$ and the total angular momentum $\vec{J}$ and by implementing interactions through an additive change of the invariant free mass operator M_0 by an interaction v (with only a dependence on the relative c.m. coordinates $\vec{p}_{\mathrm{rel}}$) which then leads to a modification of the two-particle Hamiltonian H with a resulting change of the boost $\vec{K}$ according to

$$\begin{aligned} &M = M_0 + v \qquad M_0 = 2\sqrt{\vec{p}_{\mathrm{rel}}^{\,2} + m^2} \\ &H = \sqrt{\vec{P}^2 + M^2} \\ &\vec{K} = \tfrac{1}{2}(H\vec{X} + \vec{X}H) - \vec{J} \times \vec{P}(M+H)^{-1}. \end{aligned} \tag{1}$$

The commutation relations of the Poincaré generators are maintained, provided that the interaction operator v commutes with $\vec{P}$, $\vec{X}$ and $\vec{J}$. For short-range interactions, the validity of the time-dependent scattering theory is easily established and the Moeller operators $\Omega_\pm(H, H_0)$ and the S-matrix $S(H, H_0)$ are Poincaré invariant in the sense of independence on the L-frame

$$O(H, H_0) = O(M, M_0) \qquad O = \Omega_\pm, S \tag{2}$$

[4] The asymptotic statement is mathematically precise whereas the often stated validity of N–W localization above the Compton wavelength is only 'effective'.

[5] This name was chosen in [11] in order to distinguish it from the field-mediated interactions of standard QFT.

and they also fulfil the cluster separability

$$s - \lim_{\delta\to\infty} O(H, H_0)T(\delta) \to \mathbf{1} \tag{3}$$

where the T operation applied to a two-particle vector separates the particle by an additional spatial distance δ. The subtle differences to the nonrelativistic case begin to show up for three particles [11]. Rather than adding the two-particle interactions, one has to first form the mass operators of the 1–2 pair with particle 3 as a spectator and define the 1–2 pair-interaction operator in the three-particle system

$$\begin{aligned} &M(12,3) = \left(\left(\sqrt{M(12)^2 + \vec{p}_{12}^{\,2}} + \sqrt{m^2 + \vec{p}_3^{\,2}}\right)^2 - (\vec{p}_{12} + \vec{p}_3)^2\right)^{\frac{1}{2}} \\ &V^{(3)}(12) \equiv M(12,3) - M(1,2,3) \qquad M(1,2,3) \equiv M_0(123) \end{aligned} \tag{4}$$

where the notation speaks for itself (the additive operators carry a subscript labelling and the superscript in the interaction $V^{(3)}(12)$ operators reminds us that the interaction of the two particles within a three-particle system is not identical to the original two-particle $v \equiv V^{(2)}(12)$ operator in the two-particle system). Defining in this way $V^{(3)}(ij)$ for all pairs, the three-particle mass operator and the corresponding Hamiltonian are given by

$$M(123) = M_0(123) + \sum_{i<j} V^{(3)}(ij) \qquad H(123) = \sqrt{M(123)^2 + p_{123}^2} \tag{5}$$

and lead to an L-invariant and cluster-separable three-particle Moeller operator and S-matrix, where the latter property is expressed as a strong operator limit

$$\begin{aligned} &S(123) \equiv S(H(123), H_0(123)) = S(M(123), M_0(123)) \\ &s\text{-}\lim_{\delta\to\infty} S(123)T(\delta_{13}, \delta_{23}) = S(12) \times \mathbf{1} \end{aligned} \tag{6}$$

with the formulae for other clusterings being obvious. By iteration and the use of the framework of re-arrangement collision theory (which introduces an auxiliary Hilbert space of bound fragments), this can be generalized to n-particles including bound states [12].

As in nonrelativistic scattering theory, there are many different relativistic direct particle interactions which lead to the same S-matrix. As Sokolov showed, this freedom to modify off-shell operators (e.g. $H, \vec{K}$ as functions of the single particle variables $\vec{p}_i, \vec{x}_i, \vec{j}_i$ and the interaction v) may be used to construct for each system of the above kind a 'scattering-equivalent' system in which the interaction-dependent generators $H, \vec{K}$ restricted to the images of the fragment spaces become the sum of cluster Hamiltonians (or boosts) with interactions between clusters being switched off [12, 13]. Using these interaction-dependent equivalence transformations, the cluster separability can be made manifest. It is also possible to couple channels in order to describe particle creation, but this channel coupling 'by hand' does not define a natural mechanism for interaction-induced vacuum polarization.

Even though such 'direct interaction models' between relativistic particles can hardly have fundamental significance, their very existence as relativistic theories (consistent with the physically indispensable macrocausality) helps us rethink the position of microcausal and local versus nonlocal but still macrocausal relativistic theories.

Since our intuition on these matters is notoriously unreliable and ridden by prejudices, it is very useful to have such illustrations. This is of particular interest in connection with recent attempts to implement nonlocality through noncommutativity of the spacetime substrate (see the last section). But even some old piece of QFT folklore, which claimed that the construction of unitary relativistic invariant and cluster-separable S-matrices can only be achieved through local QFT, is rendered factually incorrect (though morally correct). Direct particle interaction models may be bad physical theories, but they are useful when it comes to removing prejudices such as the idea that the cluster properties together with relativistic covariance can lead to QFT.

It turns out that if one adds crossing symmetry to the list of S-matrix properties, it is possible to show that if the on-shell S-matrix originates from a local QFT, it determines its local system of operator algebras uniquely [14]. This unicity of local algebras is of course the only kind of uniqueness which one can expect since individual fields are analogous to coordinates in differential geometry (in the sense that passing to another locally related field cannot change the S-matrix).

The new concept which implements the desired crossing property and also insures the principle of 'nuclear democracy'[6] (both properties are incompatible with the above relativistic QM!) is modular localization. In contrast to the quantum mechanical Newton–Wigner localization, it is not based on projection operators but rather is reflected in the Einstein causal behaviour of expectation values of local variables in modular-localized state vectors. Modular localization in fact relates off-shell causality, interaction-induced vacuum polarization and on-shell crossing in an inexorable manner and in particular furnishes the appropriate setting for causal propagation properties (see next section). Since it allows us to give an intrinsic definition of interactions in terms of the vacuum polarization clouds which accompany locally generated one-particle states without reference to field coordinates or Lagrangians, one expects that it serves as a constructive tool for nonperturbative investigations. This is borne out for models considered in section 5.

It is interesting to note that both localizations are realized in the Wigner theory. Used in the Bakamjian–Thomas–Coester spirit of QM of relativistic particles with the Newton–Wigner localization, it leads to relativistic invariant unitary scattering operators which obey cluster separability properties and hence are in perfect harmony with macrocausality. On the other hand, used as a starting point of modular localization one can directly pass to the system of local operator algebras without the inference of singular field coordinatization in terms of point-like fields and relate the notion of interaction (and also exceptional statistics) with microcausality and vacuum polarization clouds which accompany the local creation of one-particle states. Perhaps the conceptually most surprising result is the very different nature of the local algebras from quantum mechanical algebras.

We will present the modular localization structure of the standard half-integer spin Wigner representation in section 2.2 and that of the exceptional (anyons, massless helicity towers) representations in section 2.3. The modular theory reveals for the first time the string-like nature of the objects described by these Wigner representations.

The subject of section 3 is the functorial construction of the local operator algebras associated with the modular subspaces of the standard Wigner representations. This functorial method also applies to the Wigner helicity towers which become converted into a free string field theory. The only Wigner representation which does not permit a functorial map into operator algebras is the $d = 1 + 2$ anyon representation.

The vacuum polarization aspects of localized particle creation operators associated with exceptional Wigner representations are treated in section 4. In section 5 we explain our strategy for the construction of theories which have no real particle creation but, different from free fields, come with a rich vacuum polarization structure in the context of $d = 1 + 1$ factorizing models.

Apart from the issue of anyons, the most interesting and unexplored case of QFTs related to positive energy Wigner representations is certainly that of the massless $d = 1 + 3$

[6] Every particle may be interpreted as the bound of all others whose fused charge is the same. An explicit illustration is furnished by the bootstrap properties of $d = 1 + 1$ factorizing S-matrices [19].

'Wigner helicity towers' (called 'continuous spin' representations by Wigner). This case is in several aspects reminiscent of structures of open string theory. It naturally combines all (even, odd, supersymmetric) helicities into one indecomposable object. If it is possible to introduce interactions into this tower structure, then the standard argument of string theorist that any consistent interacting object which contains spin-2 must also contain an (at least on a quasiclassical level) Einstein–Hilbert action applies here as well[7]. Since the CCR/CAR functor carries the string-like localized Wigner wavefunction into a 'string field', i.e. an operator whose n-fold application generates an n-fold localized helicity tower, this model promises to provide an illustration of a string field theory.

Recently, there has been some interest in the problem whether the Wigner particle structure can be consistent with a noncommutative structure of spacetime where the minimal consistency is the validity of macrocausality. We will have some comments in the last section.

2. Modular aspects of positive energy Wigner representations

In the next subsection, we will briefly sketch how one obtains the interaction-free local operator algebras directly from the Wigner particle theory without passing through point-like fields. The first step is to show that there exists a relativistic localization which is different from the noncovariant Newton–Wigner localization.

2.1. Modular concepts in the scalar setting

For simplicity, we start from the Hilbert space of complex momentum space wavefunction of the irreducible $(m, s = 0)$ representation for a neutral (self-conjugate) scalar particle. In this case, we only need to remind the reader of published results [6, 16–18]:

$$\begin{aligned} H_{\mathrm{Wig}} &= \left\{ \psi(p) | \int |\psi(p)|^2 \frac{\mathrm{d}^3 p}{2\sqrt{p^2+m^2}} < \infty \right\} \\ (\mathfrak{u}(\Lambda, a)\psi)(p) &= \mathrm{e}^{\mathrm{i}pa}\psi(\Lambda^{-1}p). \end{aligned} \tag{7}$$

For the construction of the real subspace $H_R(W_0)$ of the standard $t - z$ wedge $W_0 = (z > |t|, x, y$ arbitrary) we use the $z - t$ Lorentz boost $\Lambda_{z-t}(\chi) \equiv \Lambda_{W_0}(\chi)$,

$$\Lambda_{W_0}(\chi) : \begin{pmatrix} t \\ z \end{pmatrix} \rightarrow \begin{pmatrix} \cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix} \tag{8}$$

which acts on H_{Wig} as a unitary group of operators $\mathfrak{u}(\chi) \equiv \mathfrak{u}(\Lambda_{z-t}(\chi), 0)$ and the $z - t$ reflection $j : (z, t) \rightarrow (-z, -t)$ which, since it involves time reflection, is implemented on Wigner wavefunctions by an anti-unitary operator $\mathfrak{u}(j)$ [6, 18],

$$\begin{aligned} &(\mathfrak{u}(j)\psi)(p) = \overline{\psi(-jp)} \\ &\mathfrak{u}(j)\mathfrak{u}(\Lambda)\mathfrak{u}(j)^* = \mathfrak{u}(j\Lambda j) = \mathfrak{u}(r_z(\pi)\Lambda r_z(\pi)) \\ &r_z(\pi) = \pi \text{ rotation around } z\text{-axis}. \end{aligned} \tag{9}$$

One then forms the unbounded[8] 'analytic continuation' in the rapidity $\mathfrak{u}(\chi \rightarrow -\mathrm{i}\pi\chi)$ which leads to unbounded positive operators. Using a notation which harmonizes with that of the

[7] Unfortunately, Wigner's massless helicity tower representations were dismissed as 'not used by nature' [8] before its string-like localization structure was noticed.

[8] The unboundedness of the $\mathfrak{s}$ involution is of crucial importance in the encoding of geometry into domain properties.

modular theory (see appendix A), we define the following operators in H_{Wig}:

$$\begin{aligned} \mathfrak{s} &= \mathfrak{j}\delta^{\frac{1}{2}} \\ \mathfrak{j} &= \mathfrak{u}(j) \\ \delta^{\mathrm{i}t} &= \mathfrak{u}(\chi = 2\pi t) \\ (\mathfrak{s}\psi)(p) &= \psi(-p)^*. \end{aligned} \tag{10}$$

Note that the operators which enter the definition of $\mathfrak{s}$ are functional-analytically extended geometrically defined objects within the Wigner theory; in particular the last line is the action of an unbounded involutive $\mathfrak{s}$ on Wigner wavefunctions, which involves complex conjugation as well as an 'analytic continuation' into the negative mass shell. The analyticity required here is provided by (and equivalent to) the domain of $\delta^{\frac{1}{2}} = \mathrm{e}^{\pi K}$, i.e. it is part of the functional calculus (spectral analysis) of the Hermitian boost generator K in $\delta^{\mathrm{i}t} = \mathrm{e}^{2\pi\mathrm{i}tK}$. Note that $\mathfrak{u}(j)$ is apart from a π rotation around the x-axis the one-particle version of the TCP operator. The last formula for $\mathfrak{s}$ would look the same even if we start from another wedge $W \neq W_0$. This is quite deceiving since physicists are not accustomed to consider the domain of definition as an essential part of an operator; they are rather inclined to distinguish operators only if their action on vectors leads to different expressions. If the formula describes a bounded operator it would define the operator uniquely, but in the case at hand dom $\mathfrak{s} \equiv$ dom $\mathfrak{s}_{W_0} \neq$ dom $\mathfrak{s}_W$ for $W_0 \neq W$ since the domains of δ_{W_0} and δ_W are different; in fact the geometric positions of the different W can be recovered from the $\mathfrak{s}$. These Tomita S-operators are only different in their domains but not in their formal appearance; this makes modular theory a very treacherous subject. A complete characterization would be to show that $\mathfrak{s}$ is an unbounded involution with a dense 'transparent' domain (meaning that its range is equal to its domain) and that the only distinguishing aspect is its domain of transparency which in the physics context encodes spacetime geometry.

The content of (10) is an adaptation of the spatial version of the Bisognano–Wichmann theorem to the Wigner one-particle theory [6, 18]. This theorem in turn is known to result from an application of Tomita–Takesaki modular theory to QFT. Rieffel and van Daele found a spatial generalization [20] of the operator-algebraic Tomita–Takesaki modular theory (see appendix A) which provides a general setting for a relation between real subspaces with special properties and $\mathfrak{s}$ operators with special properties. We can abstract the relevant properties in our model by observing that the anti-unitary $t - z$ reflection commutes with the $t - z$ boost $\delta^{\mathrm{i}t} = \mathrm{e}^{\mathrm{i}Kt}$, which then leads to the commutation relation $\mathfrak{j}K = -K\mathfrak{j}$ for its infinitesimal generator and hence to $j\delta = \delta^{-1}j$ involving the unbounded operator $(\delta^{\mathrm{i}})^{-\mathrm{i}} = \delta = \mathrm{e}^K$. As a result of this commutation relation and the involutive nature of the anti-unitary j, the unbounded antilinear operator $\mathfrak{s}$ is involutive on its domain of definition, i.e. $\mathfrak{s}^2 \subset 1$ with $\mathrm{Dom}(\mathfrak{s}) = \mathrm{Range}(\mathfrak{s})$, so that it may be used to define a real subspace (closed in the real sense, i.e. its complexification is not closed) as explained in the appendix. The definition of $H_R(W_0)$ in terms of +1 eigenvectors of $\mathfrak{s}$ is

$$\begin{aligned} H_R(W_0) &= \mathrm{clos}\{\psi \in H_{\mathrm{Wig}} | \mathfrak{s}\psi = \psi\} \\ &= \mathrm{clos}\{\psi + \mathfrak{s}\psi | \psi \in \mathrm{dom}\,\mathfrak{s}\} \\ \mathfrak{s}(\psi + \mathrm{i}\varphi) &= (\psi - \mathrm{i}\varphi) \qquad \psi, \varphi \in H_R(W_0). \end{aligned} \tag{11}$$

The +1 eigenvalue condition is equivalent to analyticity of $\delta^{\mathrm{i}t}\psi$ in $-\mathrm{i}\frac{1}{2} < \mathrm{Im}\, t < 0$ (and continuity on the boundary) together with a reality property relating the two boundary values on this strip. The localization in the opposite wedge, i.e. the $H_R\left(W_0^{\mathrm{opp}}\right)$ subspace, turns out to correspond to the symplectic (or real orthogonal) complement of $H_R(W_0)$ in H_{Wig}, i.e.

$$\mathrm{Im}(\psi, H_R(W_0)) = 0 \quad \Leftrightarrow \quad \psi \in H_R\left(W_0^{\mathrm{opp}}\right) \equiv \mathfrak{j}H_R(W_0) = H_R(r_z(\pi)W_0). \tag{12}$$

Furthermore, one finds the following properties for the subspaces called 'standardness':

$$H_R(W_0) + \mathrm{i}H_R(W_0) \text{ is dense in } H_{\mathrm{Wig}} \qquad H_R(W_0) \cap \mathrm{i}H_R(W_0) = \{0\}. \tag{13}$$

For completeness we sketch the proof. The closedness of the densely defined $\mathfrak{s}$ leads to the following decomposition of the domain Dom($\mathfrak{s}$):

$$\begin{aligned} \mathrm{Dom}\,(\mathfrak{s}) &= \left\{\psi \in H_{\mathrm{Wig}} | \psi = \frac{1}{2}(\psi + \mathfrak{s}\psi) + \frac{\mathrm{i}}{2}(\psi - \mathfrak{s}\psi)\right\} \\ &= H_R(W_0) + \mathrm{i}H_R(W_0). \end{aligned} \tag{14}$$

On the other hand, from $\psi \in H_R(W_0) \cap \mathrm{i}H_R(W_0)$ one obtains

$$\begin{aligned} &\psi = \mathfrak{s}\psi \\ &\mathrm{i}\psi = \mathfrak{s}\mathrm{i}\psi = -\mathrm{i}\mathfrak{s}\psi = -\mathrm{i}\psi \end{aligned} \tag{15}$$

from which $\psi = 0$ follows. In the appendix, it is shown that conversely the standardness of a real subspace H_R leads to the modular objects j, δ and $\mathfrak{s}$.

Since the Poincaré group acts transitively on the W and carries the W_0-affiliated $\mathfrak{u}(\Lambda_{W_0}(\chi))$, $\mathfrak{u}(r_{W_0})$ into the corresponding W-affiliated L-boosts and reflections, the subspaces $H_R(W)$ have the following covariance properties,

$$\begin{aligned} &\mathfrak{u}(\Lambda, a)H_R(W_0) = H_R(W = \Lambda W_0 + a) \\ &\mathfrak{s}_W = \mathfrak{u}(\Lambda, a)s_{W_0}\mathfrak{u}(\Lambda, a)^{-1} \end{aligned} \tag{16}$$

where the Poincaré transformation is only determined up to those transformations which fix the two wedges. Conversely, the modular operators $\Delta_W^{\mathrm{i}t}$ of the family of wedges generate not only the Lorentz group but also the full Poincaré symmetry [21]. It is comforting to know that the positivity of the energy implies that the inclusion of wedge spaces follows the geometric pattern of inclusions, i.e.

$$H_R(W_1) \underset{\neq}{\subset} H_R(W_2) \qquad W_1 \subset W_2. \tag{17}$$

In fact, according to the work of Borchers and Wiesbrock [21] this inclusion characterizes positive energy representations.

Having arrived at the wedge localization spaces, one may construct localization spaces for general causally complete convex spacetime regions $\mathcal{O}$ by using the fact that such regions can be obtained by intersecting wedges. The associated localization spaces should then be defined as ($\mathcal{K}$ = family of convex causally complete regions including wedges)

$$\begin{aligned} &H_R(\mathcal{O}) := \bigcap_{W \supset \mathcal{O}} H_R(W) \qquad \mathcal{O} \in \mathcal{K} \\ &H_R(\mathcal{Q}) = \bigcup_{\mathcal{O} \subseteq \mathcal{Q}} H_R(\mathcal{O}) \\ &\curvearrowright H_R(\mathcal{O}')' = H_R(\mathcal{O}). \end{aligned} \tag{18}$$

The formula in the second line in terms of inner approximations by double cones is then used as a definition for causally complete regions which are not representable as intersections of wedges, e.g. the causal disjoint of a double cone $\mathcal{O}'$. One easily checks the mutual consistency of the two definitions of which the Haag duality in the third line is a consequence. These spaces turn out to be again standard and covariant (see (19) below). According to the spatial modular theory of the appendix, such real subspaces lead to a modular s operator $\mathfrak{s}_{\mathcal{O}}$ but this time the radial and angular parts $\delta_{\mathcal{O}}$ and $\mathrm{j}_{\mathcal{O}}$ in their polar decomposition (10) cannot be described in terms of spacetime diffeomorphisms in Wigner space. To be more precise, the action of $\delta_{\mathcal{O}}^{\mathrm{i}t}$

is only local in the sense that $H_R(\mathcal{O})$ and its symplectic complement $H_R(\mathcal{O})' = H_R(\mathcal{O}')$ are transformed onto themselves (whereas $\mathfrak{j}$ interchanges the original subspace with its symplectic complement), but inside the respective regions the action of $\Delta_{\mathcal{O}}^{\mathrm{i}t}$ is 'fuzzy' or nonlocal, i.e. there is no implementing diffeomorphism[9] which renders their mathematical description more difficult. Nevertheless the modular transformations $\delta_{\mathcal{O}}^{\mathrm{i}t}$ for $\mathcal{O}$ running through all double cones and wedges generate the action of an infinite-dimensional Lie group of unitaries in the Wigner representation space. Since they are associated with real subspaces they may be thought of as being related to an infinite-dimensional geometry which in the special cases of $\mathcal{O} = W$ can be encoded into ordinary spacetime diffeomorphisms.

The geometric aspects of modular theory improve in the massless case with half-integer helicity where conformal invariance results in the conformal equivalence of double cones with wedges within the setting of the compactified Minkowski space and its covering.

The emergence of these *fuzzy acting Lie groups is a pure quantum phenomenon*; there is no analogue in classical physics. They describe hidden symmetries [22, 23] which the Lagrangian formalism does not expose and whose physical significance is not obvious.

The standardness of modular subspaces $H_R(\mathcal{O})$ can be seen from the following representation in terms of mass shell restriction of Fourier transformed test functions (rclos denotes real closure within the Wigner space):

$$H_R(\mathcal{O}) = \mathrm{rclos}\{\psi = E_m f \,|\, f \in \mathcal{D}(\mathcal{O}),\ f = f^*\} \tag{19}$$

$$(E_m f)(p) = \frac{1}{(2\pi)^2} \int f(x)\, \mathrm{e}^{\mathrm{i}px}\, \mathrm{d}^4x|_{p^2=m^2,\, p_0>0}. \tag{20}$$

$H_R(\mathcal{O}) \cap \mathrm{i}H_R(\mathcal{O}) = \{0\}$ is obvious and the denseness of $H_R(\mathcal{O}) + \mathrm{i}H_R(\mathcal{O})$ follows from a well-known analyticity argument which shows there are no nontrivial orthogonal vectors to this subspace. This dense subspace may also be characterized in terms of a closure of a space of entire functions with a Paley–Wiener asymptotic behaviour. From these representations (18), (19) it is fairly easy to conclude that the inclusion-preserving maps $\mathcal{O} \to H_R(\mathcal{O})$ are maps between orthocomplemented lattices of causally closed regions (with the complement being the causal disjoint) and modular localized real subspaces (with the symplectic or (for half-integer spin) real orthogonal complement).

The dense subspace $H(W) = H_R(W) + \mathrm{i}H_R(W)$ of H_{Wig} changes its position within H_{Wig} together with W and the same happens for the $H_R(\mathcal{O}) + \mathrm{i}H_R(\mathcal{O})$. If one takes the closure in the topology of H_{Wig} one would lose all this subtle geometric information encoded in the $\mathfrak{s}$-domains. One must change the topology in such a way that the dense subspace $H(W)$ becomes a Hilbert space in its own right. This is achieved in terms of the graph norm of $\mathfrak{s}_W$ (for the characterization of the $H_R(\mathcal{O})$ in terms of test function (19) one does not need the $\mathfrak{s}$-operator)

$$(\psi, \psi)_{G\mathfrak{s}} \equiv (\psi, \psi) + (\mathfrak{s}\psi, \mathfrak{s}\psi) < \infty. \tag{21}$$

This topology is simply an algebraic way of characterizing a Hilbert space which consists of localized vectors only. It is easy to write down a modified inner product in which the $\mathfrak{s}$ becomes a bounded operator,

$$\{\psi, \varphi\} = \left(\psi, \frac{1}{1+\delta}\varphi\right). \tag{22}$$

[9] The group generated by all these fuzzy transformations is an infinite-dimensional Lie group (with the only geometric part being the Poincaré subgroup) for which no simple description is presently known.

Clearly $\delta = \mathfrak{s}^*\mathfrak{s}$ is bounded in the new norm. This suggests to look for a more standard thermal characterization. A convenient way for doing this is to pass to the Fourier transform in the rapidity

$$f(\theta) = \frac{1}{\sqrt{2\pi}} \int \tilde{f}(\kappa)\, \mathrm{e}^{\mathrm{i}\kappa\theta}\, \mathrm{d}\kappa \tag{23}$$

and try an ansatz which is modelled on the standard thermal inner product in these Rindler variables κ,

$$(\tilde{\psi}, \tilde{\varphi})_{W,\mathrm{ther}} = \int_0^\infty \left\{ \tilde{\psi}_+^*(\kappa, p_\perp) \frac{1}{1 - \mathrm{e}^{-\beta\kappa}} \tilde{\varphi}_+(\kappa, p_\perp) + \tilde{\varphi}_-^*(\kappa, p_\perp) \frac{1}{\mathrm{e}^{\beta\kappa} - 1} \tilde{\psi}_-(\kappa, p_\perp) \right\} \mathrm{d}\kappa\, \mathrm{d}p_\perp. \tag{24}$$

This ansatz corresponds to the one-particle projection of a thermal Bose system at inverse temperature β (the $\pm$ denotes the frequency components of the wavefunction), i.e. it describes a quasi-free thermal state in a Rindler world with the role of the Hamiltonian being played by the Lorentz boost generator K in $\delta = \mathrm{e}^{-2\pi K}$. It is well known that this thermal system can be identified with the restriction of a free field ground state system to the Rindler wedge if and only if the temperature equals the Hawking temperature $T = \frac{1}{\beta} = 2\pi$. In the present setting of Wigner wavefunctions, this is a result of the fact that both inner products are obeying the same KMS condition if and only if $T = 2\pi$,

$$(\psi, \delta\varphi) = (\varphi^*, \psi^*)_{-\mathrm{i}\pi} \qquad (\tilde{\psi}, \delta\tilde{\varphi})_{\mathrm{ther}} = (\varphi^*, \psi^*)|_{+\leftrightarrow-} \tag{25}$$

i.e. the boundary value at $-\mathrm{i}\pi$ in the first formulation corresponds to the interchange of positive with negative frequencies in the manifest thermal description in the second line.

This aspect of modular localization in the Wigner one-particle theory pre-empts the fact that the associated free field theory in the vacuum state restricted to the wedge becomes thermal. We have taken localization in a wedge because then the modular Hamiltonian K has a geometric interpretation in terms of the L-boost, but the modular Hamiltonian always exists; if not in a geometric sense then as a fuzzy transformation which fixes the localization region and its causal complement. For any causally closed spacetime region $\mathcal{O}$ and its nontrivial causal complement $\mathcal{O}'$ there exists such a thermally closed Hilbert space of localized vectors and for the wedge W this pre-empts the Unruh–Hawking effect associated with the geometric Lorentz boost playing the role of a Hamiltonian (in case of ($m = 0, s =$ half-integer) representations this also holds for double cones since they are conformally equivalent to wedges).

After having obtained some understanding of modular localization, it is helpful to highlight the difference between N–W and modular localization by a concrete illustration. Consider the energy–momentum density in a one-particle wavefunction of the form $\psi_f = E_m f \in H_R(\mathcal{O})$ where supp $f \subset \mathcal{O}$, f real,

$$\begin{aligned} t_{\mu\nu}(x, \psi) &= \partial_\mu \psi_f(x) \partial_\nu \psi_f(x) + \tfrac{1}{2} g_{\mu\nu} (m^2 \psi_f(x)^2 - \partial^\nu \psi_f(x) \partial_\nu \psi_f(x)) \\ &= \langle f, c| : T_{\mu\nu}(x) : |f, c\rangle \qquad |f, c\rangle \equiv W(f)|0\rangle \end{aligned} \tag{26}$$

where on the right-hand side we use the standard field theoretic expression for the expectation value of the energy–momentum density in a coherent state obtained by applying the Weyl operator corresponding to the test function f to the vacuum. Since $\psi_f(x) = \int \Delta(x - y, m) \times f(y)\, \mathrm{d}^4 y$ we see that the one-particle expectation (26) complies with Einstein causality (no superluminal propagation outside the causal influence region of $\mathcal{O}$), but there is no way to affiliate a projector with the subspace $H_R(\mathcal{O})$ or with coherent states (the real projectors appearing in the appendix are really unbounded operators in the complex sense). We also note that as a result of the analytic properties of the wavefunction in momentum space the expectation value has crossing properties, i.e. it can be analytically continued to a matrix element of T between the vacuum and a modular localized two-particle state. This follows either by explicit computation or by using the KMS property on the field theoretic interpretation of the expectation value. A more detailed investigation shows that the appearance of this crossing (vacuum polarization) structure and the absence of localizing projectors are inexorably related. This property of the positive energy Wigner representations pre-empts a generic property of local quantum physics: *relativistic localization cannot be described in terms of (complex) subspaces and projectors, rather this must be done in terms of expectation values of local observables in modular localized states which belong to real subspaces.* This poses the question about the operational meaning of modular localization when no projectors are available. Following [24] one can characterize a modular localized state φ in $\mathcal{O}$ by the following relation to the vacuum state ω,

$$(\varphi(A) - \omega(A))^2 \leqslant c\omega([A - \omega(A)]^2) \qquad c > 0$$
$$A = A^* \in \mathcal{A}(\mathcal{O}') \tag{27}$$

i.e. this inequality should hold for all Hermitian operators from the algebra of the causal disjoint. This is necessary and sufficient for the existence of a modular localized vector $G\Omega$ where G is associated with $\mathcal{A}(\mathcal{O})$ and the domain of G^* contains $G\Omega$. For a proof we refer to [24] (section 4, lemma 4.1), see also [25].

The use of the inappropriate localization concepts[10] is the prime reason why there have been many misleading papers on 'superluminal propagation' in which Fermi's result that the classical relativistic propagation inside the forward light cone continues to hold in relativistic QFT was called into question (for a detailed critical account see [27]).

On the more formal mathematical level, this is connected to the different nature of the local algebras, in particular the absence of pure states and minimal projectors. The standard framework of QM and the concepts of 'quantum computation' simply do not apply to the local operator algebras since the latter are of von Neumann type III_1 hyperfinite operator algebras and not of the standard quantum mechanical type I. Therefore it is a bit misleading to say that local quantum physics is just QM with the nonrelativistic Galilei group replaced by Poincaré symmetry; these two requirements would lead to the kind of relativistic QM mentioned in the previous section whereas QFT is characterized by microcausality of observables and, as will be shown in the following, modular localization of states. To avoid any misunderstanding, projectors within local algebras $\mathcal{A}(\mathcal{O})$ of course exist, but they are at best able to describe fuzzy (nonsharp) localization within $\mathcal{O}$ and the vacuum is necessarily a highly entangled temperature state if restricted via this projector (in QM spatial restrictions only create isotopic representations, i.e. enhanced multiplicities, but do not cause genuine entanglement or thermal behaviour).

It is interesting that the two different localization concepts have aroused passionate discussions in philosophical circles as evidenced, e.g., from bellicose sounding title 'Reeh–Schlieder defeats Newton–Wigner' in [29]. As should be clear from our presentation, particle physics uses both, the first for causal (nonsuperluminal) propagation and the second for scattering theory where only asymptotic covariance and causality are required. As will be shown in a separate section, the modular localization in the Wigner theory has a direct functorial relation (via the CCR or CAR functor) to the local net of algebras by which the field coordinatization independent method characterizes quantum field theories (AQFT).

[10] The first relativistic localization concept for states was introduced by Licht [26]. For use in field theoretic constructions such as those in this paper, one needs the modular localization in which localization properties are encoded into domain properties of modular operators.

2.2. Generalization to (half-)integer spin

After the important remarks about the difference between the projector (or complex subspace) based localization and the relativistic causality preserving modular localization, let us return to Wigner theory and indicate how one generalizes the modular method to all (m, s) positive energy Wigner representations. The crucial observation is that all it takes to construct an involutive operator $\mathfrak{s}$ with $\mathfrak{s}^2 \subset \mathbf{1}$ and a 'transparent domain' is to have a positive energy representation of $\tilde{\mathcal{P}}_+$. Since our starting point is Wigner's irreducible positive energy representations, *the first question is how one can introduce a TCP reflection*, i.e. an antilinear operator θ which has the right commutation relation with the operators representing the connected part of the covering of the Poincaré group. The $z-t$ wedge reflections $\tilde{j}$ used in the modular theory are then equal to θ modulo a π rotation around the z-axis. The answer to this question is well known [8]: for massive half-integer spin representation, such an anti-unitary $u(\tilde{j})$ exists within the $(2s+1)$-component Wigner representation whereas for $m = 0$ one must double the one-component helicity space as to include both helicities $\pm\frac{n}{2}$. The relevant formula for the massive case is

$$\begin{aligned}(\mathfrak{u}(\theta)\psi)(p) &= D^{(s)}(\mathrm{i}\sigma_2)\overline{\psi(-\theta p)} \\ (\mathfrak{u}(\tilde{j})\psi)(p) &= D^{(s)}(\mathrm{i}\sigma_3)D^{(s)}(\mathrm{i}\sigma_2)\overline{\psi(-jp)} = D^{(s)}(\mathrm{i}\sigma_1)\overline{\psi(-jp)}\end{aligned} \tag{28}$$

with an undetermined phase factor which we have put equal to 1. The matrices $D^{(s)}$ act on $(2s+1)$-component wavefunctions; if $2s =$ even the matrices in front are ± 1. By a lengthy but straightforward computation, one checks that this operator has the expected commutation relation with the connected part of the group,

$$\begin{aligned}(\mathfrak{u}(\tilde{j})\mathfrak{u}(\tilde{\Lambda})\mathfrak{u}(\tilde{j})\psi)(p) &= D^{(s)}(\mathrm{i}\sigma_3)D^{(s)}(\mathrm{i}\sigma_2)\overline{D^{(s)}(\tilde{R}(\tilde{\Lambda}, -jp))}D^{(s)}(-\mathrm{i}\sigma_2)D^{(s)}(-\mathrm{i}\sigma_3)\psi(\Lambda^{-1}p) \\ &= D^{(s)}(\mathrm{i}\sigma_3)D^{(s)}(\tilde{R}(\tilde{r}_3(\pi)\tilde{\Lambda}\tilde{r}_3(\pi), p))\psi((r_3(\pi)\Lambda r_3(\pi))^{-1}p) \\ &= (\mathfrak{u}(\tilde{r}_3(\pi)\tilde{\Lambda}\tilde{r}_3(\pi))\psi)(p) = (\mathfrak{u}(\tilde{j}\tilde{\Lambda}\tilde{j})\psi)(p) \\ (\mathfrak{u}(\tilde{\Lambda})\psi)(p) &= D^{(s)}(\tilde{R}(\tilde{\Lambda}, p))\psi(\Lambda^{-1}p) \\ \tilde{R}(\tilde{\Lambda}, p) &= \alpha(L(p)^{-1})\alpha(\tilde{\Lambda})\alpha(L(\Lambda^{-1}p)) \qquad \alpha(L(p)) = \sqrt{\frac{p^\mu\sigma_\mu}{m}}\end{aligned} \tag{29}$$

where in the last two lines we have recalled Wigner's unitary transformation law in terms of the $\tilde{\Lambda}$- and p-dependent Wigner rotation $\tilde{R}(\tilde{\Lambda}, p) \in SU(2)$ which in turn is composed in terms of the family of boost matrices $\alpha(L(p)) \in SL(2, C)$ and the given L-transformation $\alpha(\tilde{\Lambda})$. The passing from the first to the second line corresponds to the identity

$$D^{(s)}(\mathrm{i}\sigma_1)\overline{D^{(s)}(\tilde{R}(\tilde{\Lambda}, -jp))D^{(s)}(\mathrm{i}\sigma_1)} = D^{(s)}(\tilde{R}(\tilde{j}\tilde{\Lambda}\tilde{j}, p)) \qquad \sigma_2\sigma_3 = \mathrm{i}\sigma_1. \tag{30}$$

Note that the reflection $\mathfrak{u}(\tilde{j})$ in an irreducible representation is only determined up to a phase factor; in (28) we have made a particular choice without loss of generality. Since this geometric choice may not be the same as for the Tomita $\mathfrak{j}$ in $s = j\delta^{\frac{1}{2}}$, we define the $\mathfrak{s}$-operator with a yet undetermined phase factor c as $\mathfrak{s} \equiv c\mathfrak{u}(\tilde{j})\mathfrak{u}(\mathfrak{f}_{z-t}(-\pi\mathrm{i})$ which has the desired involutive properties independent of c. A simple computation shows that the Wigner transformation of the analytically continued boost generates an additional matrix factor σ_3 which compensates that in $\mathfrak{u}(\tilde{j})$ and results in the simple formula

$$(\mathfrak{s}\psi)(p) = cD^{(s)}(\mathrm{i}\sigma_2)\psi(-p)^* \tag{31}$$

where the appearance of the negative mass shell on the right-hand side is explained in terms of the analytic continuation which goes with the definition of the unbounded positive operator $\mathrm{e}^K = \mathfrak{u}(\mathfrak{f}_{x-t}(-\pi\mathrm{i})$. Again we use this operator in order to distinguish a real subspace $H_R(W_0)$. Its angular part $\mathfrak{j} = c\mathfrak{u}(\tilde{j})$ in the polar decomposition applied to this space defines the symplectic complement $H_R(W_0)'$ in the sense of (12). Now we are ready to determine the phase factor c from a comparison between the symplectic complement $H_R(W_0)'$ and the geometric opposite $H_R(W_0')$. We have

$$\begin{aligned} \mathfrak{u}(\tilde{j})H_R(W_0) &= \mathfrak{u}(\tilde{r}(\pi))H_R(W_0) \equiv H_R(W_0') \\ H_R(W_0)' &= \mathfrak{j}H_R(W_0) = cH_R(W_0') \end{aligned} \tag{32}$$

with $\tilde{r}(\pi)$ being a π rotation around an axis perpendicular to the 3-axis. From the last relation together with the antilinearity of $\mathfrak{s}$ and the fact that $\mathfrak{s}^*$ corresponds to $H_R(W_0)'$ one has

$$c\mathfrak{s}(W_0')c^* = \mathfrak{s}^* \qquad \mathfrak{s}(W_0') \equiv \operatorname{Ad}\mathfrak{u}(r(\pi))\mathfrak{s}(W_0). \tag{33}$$

The group theoretical commutation relations of the rotation $r_z(\pi)$ with the wedge preserving boost and the reflection

$$\mathfrak{u}(r(\pi))\mathfrak{s}^* = \mathfrak{s}\mathfrak{u}(r(-\pi)) \tag{34}$$

together with the previous relations finally force c to take the following values,

$$\begin{aligned} &c^2 = \mathfrak{u}(r(2\pi)) = \begin{cases} 1 & s \text{ integer} \\ -1 & s \text{ half-integer} \end{cases} \\ &\curvearrowright c = (\mathrm{i})^{2s} \end{aligned} \tag{35}$$

where (since $-H_R(W_0) = H_R(W_0)$) without loss of generality we may select the + sign. Hence the only change in the modular theory of the $\tilde{\mathcal{P}}_+$ covering group representation is that $\mathfrak{s}(W_0') \equiv \operatorname{Ad} u(r(\pi))\mathfrak{s} = -\mathfrak{s}^*$.

Having understood the case of neutral particles, let us briefly turn to the charged case. The simplest way is to write the latter in terms of the former. To avoid lengthy notational problems, let us simply add a superscript C to the modular objects in the charged case. We define

$$\begin{aligned} &H^{(C)}_{\mathrm{Wig}} = H^{(+)}_{\mathrm{Wig}} \oplus H^{(-)}_{\mathrm{Wig}} \\ &\mathfrak{u}^{(C)}(\tilde{\Lambda})\psi^{(C)} = \mathfrak{u}(\tilde{\Lambda})\psi^{(+)} \oplus \mathfrak{u}_{\mathrm{conj}}(\tilde{\Lambda})\psi^{(-)} \\ &\mathfrak{u}^{(C)}(\tilde{j})\psi^{(C)} = D^{(s)}(\mathrm{i}\sigma_2)C\{\mathfrak{u}(\tilde{j})\psi^{(+)} \oplus \mathfrak{u}_{\mathrm{conj}}(\tilde{j})\psi^{(-)}\} \qquad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{36}$$

where the subscript conjugate denotes the complex conjugate in the matrix part of the action on wavefunctions (this affects only half-integer spins) and the $D^{(s)}(\mathrm{i}\sigma_2)$ in front is necessary in order to return to the same Lorentz transformation (36) after the charge conjugation. This leads to an $\mathfrak{s}^{(C)}$ operator,

$$\begin{aligned} &\mathfrak{s}^{(C)} = (\mathrm{i})^{2s}\mathfrak{u}^{(C)}(\tilde{j})\mathfrak{u}^{(C)}(\tilde{\Lambda}(-\pi\mathrm{i})) = \mathfrak{j}\delta^{\frac{1}{2}} \\ &H_R^{(C)}(W_0) = \left\{\psi^{(C)} = \begin{pmatrix} \varphi + \mathrm{i}\psi \\ \varphi - \mathrm{i}\psi \end{pmatrix} \middle| \varphi, \psi \in H_R(W_0)\right\} \end{aligned} \tag{37}$$

where the +1 eigenspace of $\mathfrak{s}^{(C)}$ is expressed in terms of the $\mathfrak{s}$ eigenspaces. The appearance of the factor i between $\mathfrak{u}^{(C)}(\tilde{j})$ and $\mathfrak{j}$ in the half-integer spin case (a sign can be absorbed in the definition of $\mathfrak{s}$) indicates the pre-emption of spin and statistics within the Wigner theory. The reason behind this is that this imaginary factor converts the symplectic complement into the real orthogonal complement and the latter is characteristic for the anticommutator within the field theoretic 2-pointfunction (section 3). As already noted in the scalar case, we may reconstruct the full information about the Poincaré symmetry as well as the net of localized subspaces from the relative change of domains of the Tomita involution with the change of wedges. In fact, it is fairly easy to see that a finite set of carefully chosen $\mathfrak{s}$ (6 in $d = 1 + 3$, [30]) suffices to encode the full covariance and localization information.

Finally, we have to address the important standardness problem for double cones which we define, as in the previous subsection, by intersecting wedges.

Let us now turn to the problem of showing standardness of the modular localization spaces $H_R(\mathcal{O})$. The Wigner rotation (29) contains the t-dependent 2×2 matrix factor $\alpha(L^{-1}(\Lambda^{-1}p))$ which in Pauli matrix notation reads

$$\frac{1}{\sqrt{m}}(\cosh 2\pi t \cdot p^0\mathbf{1} - \sinh 2\pi t \cdot p^1\sigma_1 + p^2\sigma_2 + p^3\sigma_3)^{\frac{1}{2}}. \tag{38}$$

In the analytic continuation in t this expression develops a square root cut in the strip $-\mathrm{i}\frac{1}{2} < \mathrm{Im}\, t < 0$. The only way to retain strip analyticity in the presence of the Wigner transformation law is to have a compensating singularity in the transformed wavefunction $\Psi\big(\Lambda_{W_0}(2\pi t)p\big)$ as t is continued into the strip. This is achieved by factorizing the Wigner wavefunction in terms of an intertwiner matrix α. Let us make the following ansatz for the original two-component Wigner wavefunction,

$$\begin{aligned} &\psi(p) = D^{(s)}(\alpha(L(p))(E_m\Phi)(p) \\ &\tilde{R}(\Lambda, p)\alpha(L^{-1}(\Lambda^{-1}p)) = \alpha(L^{-1}(p))\alpha(\tilde{\Lambda}) \end{aligned} \tag{39}$$

where in the last line we write the defining equation for the Wigner rotation as an intertwining relation between the Wigner rotation and the $SL(2, \mathbb{C})$ covering group with $\alpha(L(p))$ being the intertwiner. $\Phi_\alpha(x) \in \mathcal{D}(C), \alpha = 1, \ldots, 2s + 1$, are $(2s + 1)$-component test functions taken from a subspace of smooth test functions with support in a double cone $C \subset W_0$ and which under Poincaré transformations behave covariantly,

$$\Phi(x) \to \mathrm{e}^{\mathrm{i}pa}\alpha(\tilde{\Lambda})\Phi(\Lambda x) \qquad \alpha(\tilde{\Lambda}) \in SL(2, C). \tag{40}$$

As before in (19) $(E_m\Phi)(p)$ denotes the mass shell projection of its Fourier transform. These projections inherit the analyticity properties of the Fourier transform. For convex compact regions, these are entire functions with a Paley–Wiener–Schwartz–Hoermander C-dependent bound in imaginary direction [31]. Its intersection with the mass shell restriction leads to a complex mass shell which contains the positive as well as the negative real mass shell[11].

The covariant (undotted) spinorial transformation law[12] changes the support of $\Phi(x)$ in a geometric way. As a consequence of group theory, the spinor wavefunction (93) transforms according to

$$\psi(p) \to \alpha(\tilde{R}(\Lambda, p))\alpha(L^{-1}(\Lambda^{-1}p))(E_m\Phi)(\Lambda^{-1}p) = \alpha(L^{-1}(p))\alpha(\Lambda)\psi(\Lambda^{-1}p) \tag{41}$$

where the intertwining relation (39) was used. We see that the product ansatz $\psi = \alpha(L^{-1}(p))E_m\Phi$ maintains the strip analyticity in each factor since the intertwiner $\alpha(L^{-1}(p))$ upon transformation $p \to \Lambda(2\pi t)p$ develops a square root cut in the t-strip which compensates that of the Wigner rotation matrix whereas according to the previous remarks the $E_m\Phi$ stays analytic throughout. The W-supported test function space provides a dense set in the space of Wigner wavefunctions, i.e. $\alpha(L(p))E_m\mathcal{D}(C) \subset H_{\mathrm{Wig}}$ is dense. In the present modular context, this provides the standardness of the real subspace $H_R(C)$. This density is in fact the one-particle analogue of the Reeh–Schlieder cyclicity theorem in QFT. This factorized form of Wigner wave functions represents a dense set of $H_R(\mathcal{O})$ wave functions in terms

[11] For convex conic regions, the analytic regions in Fourier space are given in terms of dual cones which still provide sufficient analyticity in order to link the real positive with the negative mass shell within the complex mass shell [32].

[12] Since here we have to distinguish between undotted and dotted spinors, we use the notation $\alpha(\Lambda)$ and $\beta(\Lambda) = \overline{\alpha(\Lambda)}$ instead of the previous $\tilde{\Lambda}$.

of $\mathcal{O}$-supported test functions and places the analytic properties of these modular-localized subspaces into evidence.

The construction of the modular objects and modular localization subspaces in the massless case is similar as long as the helicity stays finite. According to Wigner, this is the case as long as the 'translations' of the little group of a light-like vector (the Euclidean group in two dimensions) are trivially represented. The concrete determination of the Λ, p-dependent $\tilde{R}$ requires a selection of a p-dependent family of L-transformations $L(p)$ which relate the reference vector p_R uniquely to a general p on the respective orbit. A common choice for the associated 2×2 matrices in the case of $d = 1 + 3$ is (again using the $SL(2, C)$ formalism)

$$\alpha(L(p)) = \frac{1}{\sqrt{p_0 + p_3}} \begin{pmatrix} p_0 + p_3 & p_1 - \mathrm{i}p_2 \\ 0 & 1 \end{pmatrix} \qquad m = 0 \tag{42}$$

with the associated little groups being $SU(2)$ or for $m = 0$ $\tilde{E}(2)$ (the twofold covering of the two-dimensional Euclidean group):

$$\tilde{E}(2) : \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\varphi}{2}} & z = a + \mathrm{i}b \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\varphi}{2}} \end{pmatrix} \qquad m = 0. \tag{43}$$

For the standard ((half-)integer helicity h) massless representations the 'z-translations' are mapped into the identity. As a result of the projection p_+ of the + helicity reference vector there exists a Wigner phase $\Phi(\Lambda, p)$,

$$\begin{aligned} \mathrm{p}_R\tilde{R}(\tilde{\Lambda}, p) &= \tilde{R}(\tilde{\Lambda}, p)_{11} = \mathrm{e}^{\frac{1}{2}\mathrm{i}\Phi(\tilde{\Lambda}, p)} \\ \tilde{R}(\tilde{\Lambda}, p) &= \alpha(\tilde{L}^{-1}(p))\tilde{\Lambda}\alpha(\tilde{L}(\Lambda^{-1}p)). \end{aligned} \tag{44}$$

The irreducible Wigner wavefunctions of helicity $h = 0, \pm\frac{1}{2}, \pm 1, \ldots$ transform under L-transformation simply as

$$\psi(p) \to \mathrm{e}^{\mathrm{i}h\Phi(\Lambda, p)}\psi(\Lambda^{-1}p). \tag{45}$$

The modular localization aspects of the integer helicity case follow similar steps as in the massive case; the difference of the 'boost' family (42) and the Wigner 'rotation' (little group) (43) as compared to the massive case does not affect the core of the arguments. It is evident (and well known [8]) that although the irreducible representation is one dimensional, one needs the opposite helicity representation in addition, in order to represent the TCP transformations and the W_0-affiliated reflection $\mathfrak{u}(\tilde{j})$; there is no other possibility to change the sign of the Wigner phase back to its original value. The calculation can be simplified by noting that it is sufficient to check the correctness of the commutation relation $\mathfrak{u}(\tilde{j})\mathfrak{u}(\tilde{\Lambda})\mathfrak{u}(\tilde{j}) = \mathfrak{u}(\tilde{j}\tilde{\Lambda}\tilde{j}) = \mathfrak{u}(\tilde{r}_z(\pi)\tilde{\Lambda}\tilde{r}_z(\pi))$. Since the computations in the doubled Wigner space containing both helicities are analogous, we will only quote the result

$$\begin{aligned} &H^{(d)}_{\mathrm{Wig}} = H_+ \oplus H_- \\ &(\mathfrak{u}(\tilde{j})\psi)(p) = \sigma_1\psi(-jp) \\ &(\mathfrak{s}\psi)(p) \equiv c\mathfrak{u}(\tilde{j})\mathfrak{u}(\tilde{\Lambda}(-\mathrm{i}\pi)) = c(\iota\sigma_2)\overline{\psi(-p)}. \end{aligned} \tag{46}$$

The pre-factor c which accounts for the mismatch between the geometric/symplectic opposite is again 1, i in the integer respectively half-integer helicity case which for half-integer helicities expresses the mismatch between the geometric and symplectic opposite.

The standardness of the double cone localization may be done by restoring the $(2s + 1)$-component formalism by returning to the situation before the projection (44) and introducing intertwiners from a $(2s + 1)$-component test function space to a $2s + 1$ Wigner space.

There is however a more elegant way of implementing modular theory by the use of the fact that the Poincaré covariance of the Wigner theory allows an extension (without enlarging the representation space) to the 15-parametric conformal covariance on the Dirac–Weyl compactified Minkowski space, respectively its double covering. This covariance allows us to transport the modular theory of the standard wedge W_0 directly to that of double cones [33].

It is also fairly easy to see that the modular formalism works for half-integer spin in $d = 1 + 2$ dimensions. In the massive case, one adjusts the 2×2 matrix formalism so that the rotation subgroup (the little group of $p_R = (m, 0, 0)$) is diagonal, i.e. one chooses the $SL(1, 1)$ description of the Lorentz group covering $\tilde{\mathcal{L}}$,

$$\mathbf{p} = p^{\mu}\hat{\sigma}_{\mu} \qquad \hat{\sigma}_{\mu} : 1, \sigma_1, \sigma_2 \tag{47}$$

$$\begin{aligned}
&\mathbf{p} \rightarrow \mathbf{p} = \alpha(\tilde{\Lambda})\mathbf{p}\alpha(\tilde{\Lambda})^* \qquad \alpha(\tilde{\Lambda}) = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \qquad |a|^2 - |b|^2 = 1 \\
&\curvearrowright \alpha(\tilde{\Lambda}) = \frac{1}{\sqrt{1-\gamma\bar{\gamma}}} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{1}{2}\omega} & \gamma \mathrm{e}^{\mathrm{i}\frac{1}{2}\omega} \\ \bar{\gamma}\mathrm{e}^{-\mathrm{i}\frac{1}{2}\omega} & \mathrm{e}^{-\mathrm{i}\frac{1}{2}\omega} \end{pmatrix} \qquad |\gamma| < 1 \\
&\alpha(L(p)) = \sqrt{\mathbf{p}} = \frac{1}{\sqrt{2m(p^0+m)}} \begin{pmatrix} p^0 + m & p^1 - \mathrm{i}p^2 \\ p^1 + \mathrm{i}p^2 & p^0 - m \end{pmatrix}.
\end{aligned} \tag{48}$$

As a reference wedge W_0 for the modular theory, we choose the $x - t$ wedge. The spin s representation of the little group ($\gamma = 0$)

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{1}{2}s\varphi} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{1}{2}s\varphi} \end{pmatrix} \tag{49}$$

suggests that the extension to an anti-unitary reflection $\mathfrak{u}(\tilde{j})$ requires (as in the massless $d = 1 + 3$ case with half-integer helicity) the presence of two components. However, this turns out to be wrong as a result of a subtle point in $d = 1 + 2$, namely the existence of a parity operator P which changes the sign of y as well as that of the phase φ. The resulting modular objects acting on one-component wavefunctions are

$$\begin{aligned}
(\mathfrak{u}(\tilde{j})\psi)(p) &= \overline{\psi(-jp)} \\
(\mathfrak{s}\psi)(p) &= (\mathfrak{j}\mathfrak{u}(\Lambda_{t-x}(-\mathrm{i}\pi))\psi)(p) \qquad \mathfrak{j} = c\mathfrak{u}(\tilde{j}) \\
&= c\overline{\psi(-p)} \qquad c = \begin{cases} \mathrm{i} & s \text{ half-integer} \\ 1 & s \text{ integer.} \end{cases}
\end{aligned}$$

This is the form of the Tomita involution $\mathfrak{s}$ for neutral particles. The doubling in the presence of antiparticles $\mathfrak{s} \rightarrow \mathfrak{s}^{(C)}$ has been explained before. The standardness for the double cone intersections $H_R(\mathcal{O}) = \cap_{W \supset \mathcal{O}} H_R(W)$ is again based on the use of the intertwiner $\alpha(L(p))$ and test function spaces with the support region $\mathcal{O}$. There is nothing new to learn from the adaptation of the above proof to $d = 1 + 2$.

The $d = 1 + 2$ zero mass representations have a little group which is the one-dimensional Euclidean group, i.e. it consists of 'translations' only. Those with a nontrivial representation of this little group belong to the exceptional cases of the next section. The only remaining representation is that of a scalar massless field whose modular theory is entirely similar to the scalar massless $d = 1 + 3$ case.

2.3. Exceptional cases: anyons and infinite 'spin towers'

All positive energy representations admit an extension from a $\tilde{\mathcal{P}}_+^{\uparrow}$ to $\tilde{\mathcal{P}}_+$ by an anti-unitary $t - x$ reflection $\mathfrak{u}(j)$ which commutes with the $t - x$ L-boost. This insures the existence of a Tomita modular operator $\mathfrak{s}$ for the standard wedge and, as a consequence of covariance, the existence net of W-localized real subspaces of H_{Wig}. For the Wigner representations of the previous section, this entails the standardness (nontriviality and denseness) of the modular localization spaces of arbitrarily small double cones obtained by forming nontrivial intersections of wedges. This (as will be seen in what follows) is not possible in the case of the exceptional representations treated in this section. As will be shown, their compact localization spaces $H_R(\mathcal{O})$ are trivial. In the transition to QFT, this corresponds to the statement that there are no operators whose one-time application to the vacuum generates double cone $\mathcal{O}$-localized Wigner states.

However the operator algebras obtained functorially from the Wigner theory (as described in the next section) may have inner symmetries[13] and could still contain a compactly localizable invariant ('neutral') subalgebra which would be a candidate for an observable algebra. This situation is expected in the case of 'free' anyons where the charge-carrying operators are those which obey space-like braid group statistics commutation relations whereas the neutral operators should be commutative for space-like separation of their localizations.

The special role of $d = 1 + 2$ spacetime dimensions for the existence of braid group statistics is due to the fact that the universal covering group is infinite sheeted and not twofold as considered in the previous section. The best way to obtain a parametrization for any spin $s \in R_+$ ('anyons') is to use the Bargmann [34] parametrization

$$\{(\gamma, \omega) \mid \gamma \in \mathbb{C}, |\gamma| < 1, \omega \in \mathbb{R}\} \tag{50}$$

where the notation is that of (48). From the matrix multiplication for the twofold covering (48) it is easy to read the composition law for the universal covering,

$$\begin{aligned}
&(\gamma_2, \omega_2)(\gamma_1, \omega_1) = (\gamma_3, \omega_3)\\
&\gamma_3 = \left(\frac{1 + \gamma_2\bar{\gamma}_1\, \mathrm{e}^{-\mathrm{i}\omega_1}}{1 + \gamma_2\gamma_1\, \mathrm{e}^{-\mathrm{i}\omega_1}}\right)\\
&\curvearrowright \omega_3 = \omega_1 + \omega_2 + \frac{1}{\mathrm{i}} \log \left\{\frac{1 + \gamma_2\bar{\gamma}_1 \mathrm{e}^{-\mathrm{i}\omega_1}}{1 + \bar{\gamma}_2\gamma_1 \mathrm{e}^{\mathrm{i}\omega_1}}\right\}.
\end{aligned} \tag{51}$$

From these composition laws, one may obtain the irreducible transformation law of a (m, s) Wigner wavefunctions in terms of a one-component Wigner wavefunction representation involving a Wigner phase $\varphi((\gamma, \omega), p)$.

Different from the covering group situation in the case of the conformal group where the covering structure is reflected in a covering space of the compactified Minkowski space, there is no way to naturally encode the covering aspect of the $d = 1 + 2$ Poincaré group $\tilde{\mathcal{P}}$ into the underlying spacetime. There exists however a ramified spatial covering which creates a multisheetedness for space-like cones (whose geometric cores are straight semi-infinite space-like strings). A space-like cone $\mathcal{C}$ can be translated so that its apex starts at the origin and the direction can be recorded by any point in its intersection r with the space-like unit hyperboloid (two-dimensional de Sitter spacetime). Since the latter has an infinite sheeted space-like covering, the pairs (C, r) are able to 'sense' the action of the covering group on a wedge W or a space-like cone C after we introduce a reference pair (C, r_0) which plays a similar role to a cut in complex function theory. Such constructions are well known from studies of braid group statistics in $d = 1 + 2$ QFT [35].

It is easy to see that the compactly localized spaces are trivial $H_R(\mathcal{O}) = \{0\}$. By a Poincaré transformation, one can always place a double cone at zero so that it is symmetric

[13] Even in case of neutral particles such as Majorana fermions there are remaining discrete inner symmetries.

with respect to rotations around zero. In that case, the modular $\mathfrak{s}_{\mathcal{O}}$ commutes with the rotation. A 2π rotation on a $H_R(\mathcal{O})$ wavefunction however yields a complex phase factor $e^{i2\pi s}$ times the original wavefunction which is compatible with the antilinear nature of $\mathfrak{s}_{\mathcal{O}}$ only for $s = \frac{n}{2}$, i.e. by use of the spin-statistics connection only for bosons/fermions.

The general structure of modular theory for positive energy representations insures the standardness of the wedge spaces. But only by doing explicit intersection calculations for $s \neq \frac{n}{2}$ can one decide the standardness of space-like cone localizations. For this purpose, one defines the subset of the Poincaré group manifold $S(C) = \left\{g \in \mathcal{P}_+^\uparrow \,\middle|\, gW_0 \supset C\right\}$ and looks for partial intertwiners $u_{S(C)}$ which allow us to find a dense set of wavefunctions of the form $u_{S(C)}E_m\Phi$ in terms of mass shell restrictions of Fourier transforms of C-supported test functions. Such partial intertwiners were recently successfully constructed by Mund [25]. This solves the standardness of $H_R(C)$, i.e. the nontriviality of modular localization in space-like cones.

There is only one remaining exceptional case, namely the somewhat mysterious $d \geqslant 1+3$ massless 'infinite helicity tower'[14]. These cases also resisted a Lagrangian quantization. There even exists a mathematical theorem which states that such representations cannot occur as subrepresentations within a Wightman setting of point-like fields [37]. However the modular localization method can deal with this more general situation.

The helicity tower representation in $d = 1+3$ results from a faithful unitary representation V of the twofold covering of the two-dimensional Euclidean group $\tilde{E}(2)$,

$$\begin{aligned}
&(\mathfrak{u}(a, \tilde{\Lambda})\psi)(p) = V(\alpha(L^{-1}(p))\Lambda\alpha(L(\Lambda^{-1}p)) \cdot \psi(\Lambda^{-1}p) \\
&(V_{\kappa,\pm}(\tilde{E}(\varphi, z))f)(\theta) = e^{i\kappa|z|\cos(\operatorname{Arg} z-\theta)+i\frac{(1-\varepsilon)}{2}\frac{1}{2}\varphi} f(\theta-\varphi) \qquad \varepsilon = \pm \\
&\tilde{E}(2) : \tilde{E}(\varphi, z) \equiv \begin{pmatrix} e^{i\frac{\varphi}{2}} & z \\ 0 & e^{-i\frac{\varphi}{2}} \end{pmatrix} \qquad z = a_1 + ia_2.
\end{aligned} \tag{52}$$

Here the inequivalent irreducible representations of $\tilde{E}(2)$ are characterized by two numbers, $\kappa \geqslant 0$ is the value of the Euclidean translations in $\tilde{E}(2)$ and the sign of $\varepsilon = \pm$ corresponds to (half)integer Euclidean angular momentum. The dot in the first line is meant to indicate that the values of the wavefunction ψ are in another Hilbert space $\mathfrak{h}$ on which V acts; it can be realized as an L^2 integrable space of functions on the circle on which V acts according to the second formula. Introducing a basis $e^{in\theta}$, $n = 0, \pm 1, \ldots,$ in which the angular action is diagonal, the wavefunction space consists of an infinite-component vector of functions $\{\psi_n(p)\}$. It is remarkable that this zero mass representation does not share the property of scale (and conformal) invariance. The finite helicity representations of the previous subsection admit a fixed-point group which is actually somewhat larger than $\tilde{E}(2)$, namely it permits an extension by

$$\begin{aligned}
&D_\lambda \operatorname{Dil}(\lambda), D_\lambda = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \\
&D_\lambda \operatorname{Dil}(\lambda)p_R = p_R \qquad p_R = \left(\tfrac{1}{2}, 0, 0, \tfrac{1}{2}\right).
\end{aligned} \tag{53}$$

The combination of dilation with a $t - z$ Lorentz transformation keeps the reference point fixed and does not change the κ value of the $\tilde{E}(2)$ representation if and only if the Euclidean translation is trivially represented (i.e. if $\kappa = 0$). The enlargement of the little group through

[14] In order to highlight its relation to string theory, we have chosen this terminology instead of Wigner's 'continuous spin'.

the use of the dilation in (53) is the reason why without enlarging the representation space the symmetry group extends from the Poincaré group to the conformal group. Usually, the classification of irreducible representations of the conformal group is done [36] without using the possibility of enlargement of the little group for $\kappa = 0$. This extension argument breaks down for $\kappa \neq 0$. The presence of the 'translation' parameter z in the Wigner operator V in (52) weakens the analytic properties and raises the suspicion that the double cone localization spaces $H_R(\mathcal{O})$ are trivial. Indeed, the result of Yngvason [37] shows that $\kappa \neq 0$ representation cannot occur as irreducible components in a QFT generated by point-like fields. On the other hand, Brunetti *et al* [40] recently arrived at the surprising result that all positive energy Wigner representations have nontrivial space-like cone localization spaces $H_W(\mathcal{C})$ in all spacetime dimensions[15]. The existence of (open, semi-infinite, space-like) strings in Wigner's fundamental classification of positive energy Poincaré group representations is somewhat of a surprise since the point-like/string-like nature of the representation-generating objects is not an input but rather a consequence of more basic physical principles; for the first time, one is confronting 'natural' strings which exist in every spacetime dimension $d \geqslant 1+2$. Although the terminology 'string' refers primarily to their best possible localization, they also enjoy for $d > 1+2$ the 'stringiness' of strings in string theory, namely the presence of spin/helicity towers (which according to a popular belief make string theories useful for describing helicity $h = 2$ gravity). The important remaining problem is to construct string-like generators whose test function smearing, in analogy with point-like fields, generates the spaces $H_R(\mathcal{C})$ according to

$$H_R(\mathcal{C}) = \overline{\{u(p,\mathcal{C})E_m f \,|\, \operatorname{supp} f \subset \mathcal{C}\}} \tag{54}$$

i.e. to find intertwiner operators which transform infinite-component (helicity-indexed) test functions into $\mathcal{C}$-localized Wigner wavefunctions. Whereas in the standard case these localization intertwiners are the adjoints of those which covariantize the Wigner canonical description [8], this ceases to be so for string-like localization; for this reason covariantization attempts [37, 38] did not reveal anything about the nature of the optimal localization, apart from the fact that it cannot be point-like. The analytic properties which modular localization imposes on the intertwiners result from the requirement that they intertwine the Wigner cocycles V with cocycles with better analyticity properties. Having arrived at an explicit formula for the spaces $H_R(\mathcal{C})$, one only needs to apply the Weyl functor (in the case of integer helicities) in order to convert the one-string spaces into a full string field theory. We defer the relevant computations to forthcoming work.

In the rest of this section, we comment on some peculiarities of the little group structure for massless theories in $d = 1+2$. Different from the higher dimensional infinite-component helicity towers this representation is one dimensional. If we copy the previous formula by changing E_2 to E_1 we obtain

$$\begin{aligned} (\mathrm{u}(\tilde{\Lambda})\psi)(p) &= \mathrm{e}^{\mathrm{i}\kappa z(\Lambda,p)}\psi(\Lambda^{-1}p) \\ V_\kappa(E_1(z)) &= \mathrm{e}^{\mathrm{i}\kappa z(\Lambda,p)} \qquad z \in R \\ E(1) : E(z) &= \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{55}$$

In writing the matrix representation for the Euclidean translation, we have used the fact that the twofold covering $SL(2,R)$ description of the $d = 1+2$ Lorentz group is included in the previous $SL(2,C)$ formalism by omitting the imaginary σ_2-matrix and therefore the little group $E(1)$ is the real subgroup of $E(\varphi, z)$ (which forces z to be real). The physical

[15] In a first version of the present work, we had the weaker result of $d = 1+2$ space-like cone localization for $\kappa \neq 0$.

interpretation is most clearly seen by computing the action of $E(z)$ on spacetime coordinates ($\check{\sigma}$: omission of σ_2),

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} x^{\mu}\check{\sigma}_{\mu} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = x'^{\mu}\check{\sigma}_{\mu}$$
$$\curvearrowright \begin{pmatrix} x'^0 \\ x'^1 \\ x'^3 \end{pmatrix} = \begin{pmatrix} 1+\frac{z^2}{2} & z & -\frac{z^2}{2} \\ z & 1 & -z \\ \frac{z^2}{2} & z & 1-\frac{z^2}{2} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^3 \end{pmatrix} \tag{56}$$
$$\begin{pmatrix} 1+\frac{z^2}{2} & z & -\frac{z^2}{2} \\ z & 1 & -z \\ \frac{z^2}{2} & z & 1-\frac{z^2}{2} \end{pmatrix} = \mathrm{Rot}(-2\varphi)L(-\chi,\varphi)$$

$$L(-\chi,\varphi) = \mathrm{Rot}(\varphi)L(-\chi)\,\mathrm{Rot}(-\varphi) \qquad \tan\varphi = \frac{z}{2} \quad \sinh\chi = z\sqrt{1+\frac{z^2}{2}}. \tag{57}$$

The last line in (56), which follows by straightforward calculations with 3×3 matrices, has the following interpretation. First, one performs a Lorentz boost $L(-\chi,\varphi)$ in a direction $e(\varphi) = \mathrm{Rot}(\varphi)e_1$ in the 1–3 plane. This changes the light-like vector $e_0 + e_3$, but by a -2φ rotation one can bring it back into its original position. The change suffered by e_1 can be described as follows: the fixed light-like vector and e_1 together span a lightfront plane and the $E(z)$ transformation acts in that plane by transforming the e_1; in fact, this planar transformation is a Galilei transformation in which the time is the affine light ray parameter of that lightfront. It should not come as a surprise that these Euclidean translations of the Wigner little group play a prominent role in the lightfront holography. Following Yngvason [37] one notes that the argument that there exists an $SL(2,R)$ automorphism (53) of $E(1)$,

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & \lambda^2 z \\ 0 & 1 \end{pmatrix}. \tag{58}$$

The existence of this automorphism which relates the irreducible representation $V_{\kappa}(\cdot)$ to the inequivalent $V_{\lambda^2\kappa}(\cdot)$ is the crucial step in proving that these representations cannot occur in a point-like setting. A direct proof of the triviality of compact localization based on contradiction of the transformation law $E(1)$ with the existence of a double cone localized space $H_R(\mathcal{O})$ for a rotational symmetric $\mathcal{O}$ will be given in a separate paper. On the other hand, the standardness of the space-like cone spaces $H_R(\mathcal{C})$ is covered by the BGL theorem [18] which is valid for all bosonic/fermionic positive energy representations.

A pedestrian way to see that there can be no compact localization for the case at hand for $d = 1+2$, $\kappa \neq 0$ may be given along the following lines. Start with a wedge in $t - x_3$ direction where the modular theory gives a Wigner wavefunction $\psi(p)$ which is the boundary value of a function analytic in the boost rapidity $-\mathrm{i}\pi < \chi < 0$ and obeys the $H_R(W_0)$ relation $\mathfrak{s}\psi(p) \equiv \overline{\psi(-p)} = \psi(p)$ where the two mass shells have been linked by a complex χ-path followed by a $\mathfrak{u}(j)$ reflection. The opposite wedge would have the opposite analytic behaviour. Combining both requirements gives a periodic function which because it is also required to be continuous and square integrable and hence according to the Liouville theorem must vanish. This of course was expected since we know from modular theory that $H_R(W)\cap H_R(W') = \{0\}$. But now perform a shift which pushes the two wedges against each other so that they intersect in the region $|x_3| < \frac{a}{2}$. In that case, the modular equation for the intersection which is the causal complement of this region is a quasiperiodicity relation in which the value on one rim of the slab is related to the opposite rim by a factor $\mathrm{e}^{\mathrm{i}p_3 a}$. This defines a nontrivial function space of entire functions in the p_3 variable with the complex Paley–Wiener bound which corresponds to the x_3-localization of the intersection. By the same token, one obtains a space of Paley–Wiener entire functions in the p_1-variable. The intersection for this special geometric situation has an envelope of holomorphy [39] which consists of entire functions in two variables with Paley–Wiener bounds. This intersection space has the $\frac{\pi}{2}$ rotation as a symmetry transformation. But this rotation generates through the transformation law (55) a p-dependent exponential $\mathrm{e}^{\mathrm{i}\kappa z(\mathrm{rot}(\pi),p)}$ which leads to an inconsistency with the Paley–Wiener bound. Therefore, the conclusion is that the intersection space is trivial. This contradiction would not have arisen in the standard case because in that case the Wigner rotation only contributes a power correction in p coming from the p-dependence of the Wigner rotation factor.

If we now also admit anyonic spin in the $d = 1 + 2$, $m = 0$ situation, we are in for a small surprise. The Wigner little group in the universal covering $\widetilde{SL(2,R)} \simeq \widetilde{SU(1,1)}$ is a tiny bit bigger than within $SL(2, R)$, namely it consists of

$$E_{\mathrm{red}}(1) \times \mathbb{Z} \tag{59}$$

where the meaning will be clear in a moment. Let us first rewrite $E(1)$ in terms of the $SU(1, 1)$ formalism,

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i} & 1 \\ \mathrm{i} & -1 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} -\mathrm{i} & -\mathrm{i} \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1+\mathrm{i}\frac{z}{2} & -\mathrm{i}\frac{z}{2} \\ \mathrm{i}\frac{z}{2} & 1-\mathrm{i}\frac{z}{2} \end{pmatrix} \\
&\qquad = \frac{1}{\sqrt{1-\gamma\bar{\gamma}}}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{1}{2}\omega(z)} & \gamma(z)\,\mathrm{e}^{\mathrm{i}\frac{1}{2}\omega(z)} \\ \bar{\gamma}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}\omega(z)} & \mathrm{e}^{-\mathrm{i}\frac{1}{2}\omega(z)} \end{pmatrix} \\
&\qquad \mathrm{e}^{\mathrm{i}\frac{1}{2}\omega(z)} = \left(\frac{1+\mathrm{i}\frac{z}{2}}{1-\mathrm{i}\frac{z}{2}}\right)^{\frac{1}{2}} \qquad \gamma(z) = \frac{-\mathrm{i}\frac{z}{2}}{1+\mathrm{i}\frac{z}{2}}.
\end{aligned} \tag{60}$$

In this way of writing, it is easy to pass to the $\widetilde{SU(1,1)}$ Bargmann formalism (51)

$$\begin{aligned}
&(\omega(z) + 2\pi n, \gamma(z)) \\
&\widetilde{E(1)} = \mathbb{R} \times \mathbb{Z} \\
&V_{\kappa,s}(z,n) = \mathrm{e}^{\mathrm{i}\kappa z}\mathrm{e}^{\mathrm{i}sn}
\end{aligned} \tag{61}$$

where we have added a discrete term which is the only modification that does not affect the fixed point condition. Therefore these massless objects are 'stringy' for two reasons: on the one hand, compact localization is excluded because of an anyonic phase factor and on the other hand, even in the case where these objects only pick up a sign under 2π rotation, the phase factor from the $E(1)$ translation would still prevent a compact localization.

3. From Wigner representations to the associated local quantum physics

In the following, we will show that such a net of operator algebras of free particles with half-integer spin/helicity can be directly constructed from the net of modular localized subspaces in standard Wigner representations. For integral spin s one uses the Weyl functor for the definition of the local subalgebras in Fock space [15, 16, 18],

$$\mathcal{A}(\mathcal{O}) \equiv \mathrm{alg}\{\mathrm{Weyl}(H_R(\mathcal{O}))|\psi \in H_R(\mathcal{O})\}. \tag{62}$$

Here the reader should recall that the Weyl functor Weyl(·) is a map Γ from Wigner wavefunctions to unitary operators in Fock space: in terms of particle/antiparticle creation/annihilation operators, one has

$$
\begin{aligned}
&H_{\mathrm{Wig}} \xrightarrow{\Gamma} B(H_{\mathrm{Fock}}) : \psi \to \mathrm{Weyl}\,(\psi) = \mathrm{e}^{\mathrm{i}C^{sa}(\psi)} \qquad \psi \in H_{\mathrm{Wig}} \\
&C^*(\psi) = \sum_{s_3=-s}^{s} \int (a^*(p,s_3)\psi^{(a)}_{s_3}(p) + b(p,s_3)\overline{\psi^{(b)}_{s_3}(p)})\frac{\mathrm{d}^3 p}{2\omega} \\
&\quad \equiv \int (a^*(p) \quad b^*(p)) \begin{pmatrix} \psi^{(a)}(p) \\ \overline{\psi^{(b)}(p)} \end{pmatrix} \frac{\mathrm{d}^3 p}{2\omega}
\end{aligned}
\tag{63}
$$

where C^{sa} denotes the self-adjoint combination $\frac{1}{\sqrt{2}}(C + C^*)$ and in the last line we use the previous vector notation. The formula refers to the Wigner theory only; point-like fields or covariant smearing functions are not entering here.

The wavefunctions in the definition $\mathcal{A}(\mathcal{O})$ belonging to the subspace $H_R(\mathcal{O})$ obey the modular restriction of the previous section, i.e. the strip analyticity and the complex conjugate relation between their boundary values linking the complex conjugate antiparticle wavefunction on the backward mass shell to the particle wavefunction through analytic continuation via the modular formalism of the previous section.

The local net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{K}}$ may be obtained in two ways: either one first constructs the spaces $H_R(\mathcal{O})$ via (18) and then applies the Weyl functor, or one first constructs the net of wedge algebras (63) and then intersects the algebras according to

$$
\mathcal{A}(\mathcal{O}) = \bigcap_{W\supset\mathcal{O}} A(W). \tag{64}
$$

The proof of the net properties follows from the well-known theorem that the Weyl functor relates the orthocomplemented lattice of real subspaces of H_{Wig} (with the complement H_R' of H_R being defined in the symplectic sense of the imaginary part of the inner product in H_{Wig}) to that of von Neumann subalgebras $\mathcal{A}(H_R) \subset \mathcal{B}(H_{\mathrm{Fock}})$ (with the complement being the von Neumann commutant) [41]. The geometric aspects of the modular localization in the Wigner theory in terms of the lattice of causally complete regions $\mathcal{O} \in \mathcal{K}$ finally relate the spacetime causality structure with the lattice structure of von Neumann algebras and their commutants.

In order to present the relation between the group theoretical and modular aspects of Wigner representation spaces and the Tomita–Takesaki modular theory of operator algebras, we need to explain some basic notions of the latter. Its main content can be focussed into two formulae,

$$
\begin{aligned}
&SA\Omega = A^*\Omega \qquad \curvearrowright S = J\Delta^{\frac{1}{2}} \qquad A \in \mathcal{A} \\
&\mathrm{Ad}\,\Delta^{\mathrm{i}t}\mathcal{A} = \mathcal{A} \qquad \mathrm{Ad}\,J\,\mathcal{A} = \mathcal{A}'.
\end{aligned}
\tag{65}
$$

The first line is a definition of an unbounded operator S relative to a state vector Ω in terms of the Hermitian conjugation in an operator algebra $\mathcal{A}$. Since 'standardness' of the pair[16] $(\mathcal{A}, \Omega)$ is assumed, the operator S has analogous properties to the $\mathfrak{s}$ of the one-particle Wigner theory, it is closable and hence it permits a polar decomposition (as before we retain the same notation for its closure) as well as antilinear, involutive and transparent on its domain ($\mathrm{Dom}\, S = \mathrm{Range}\, S$) which consists of vectors of the form $G\Omega$ with G affiliated with $\mathcal{A}$; the real +1 eigenspace of S is the real closure of $\mathcal{A}^{sa}\Omega$ formed with the self-adjoint part of the algebra. The nontrivial part of the Tomita theorem is contained in the second line: the modular unitary $\Delta^{\mathrm{i}t}$ defines an automorphism of the algebra $\mathcal{A}$ (which turns out to be dependent only on the state and not on the implementing vector) and an anti-unitary modular involution J whose Ad-action maps $\mathcal{A}$ into its von Neumann commutant $\mathcal{A}'$.

[16] In the operator-algebraic context, standardness means Ω is cyclic ($\overline{\mathcal{A}\Omega} = H$) and separating ($A\Omega = 0 \curvearrowright A = 0$).

The noted analogy goes much deeper, in that the algebraic modular theory of the local algebras $\mathcal{A}(W)$ obtained via the Weyl functor is in fact the functorial image of the spatial modular objects under the same functor Γ,

$$J, \Delta, S = \mathfrak{d}(\mathfrak{j}, \delta, \mathfrak{s}). \tag{66}$$

Using the intuitive notation from the setting of coherent states, we define

$$\begin{aligned} &h \in H \xrightarrow{\Gamma} \mathrm{e}^{\mathrm{i}h} \in \mathcal{H} = \mathrm{e}^{\mathrm{i}H} \qquad (\mathrm{e}^{\mathrm{i}h}, \mathrm{e}^{\mathrm{i}k}) = \mathrm{e}^{(h,k)} \\ &\mathrm{Weyl}(h)\,\mathrm{e}^{\mathrm{i}k} = \mathrm{e}^{-\frac{1}{2}(h,h)}\,\mathrm{e}^{-(h,k)}\,\mathrm{e}^{\mathrm{i}(h+k)} \curvearrowright \mathrm{Weyl}(h)\,\mathrm{Weyl}(k) \\ &\qquad\qquad = \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}(h,k)}\,\mathrm{Weyl}(k)\,\mathrm{Weyl}(h) \end{aligned} \tag{67}$$

and extend the Γ map to linear and antilinear operators $\mathfrak{a}$ in H,

$$\mathrm{e}^{\mathfrak{a}}\,\mathrm{e}^{\mathrm{i}h} \equiv \mathrm{e}^{\mathrm{i}\mathfrak{a}h} \qquad \mathrm{e}^{\mathfrak{j}}\,\mathrm{e}^{\mathrm{i}h} \equiv \mathrm{e}^{-\mathrm{i}\mathfrak{j}h}.$$

Since the coherent states form a total set, the Γ-operation leads to well-defined operators in the Fock space $\mathcal{H}$. We now claim

$$S = \mathrm{e}^{\mathfrak{s}} = \mathrm{e}^{\mathfrak{j}}\,\mathrm{e}^{\delta^{\frac{1}{2}}} \tag{68}$$

where on the left-hand side there appears the above Tomita S-operator of the operator algebra theory and on the right-hand the functorial image of the geometrically defined $\mathfrak{s}$ in the Wigner theory. The proof is as follows:

$$\begin{aligned} \mathrm{e}^{\mathfrak{s}}\,\mathrm{Weyl}(h)\,\mathrm{e}^{0} &= \mathrm{e}^{\mathfrak{s}}\,\mathrm{e}^{-\frac{1}{2}(h,h)}\,\mathrm{e}^{\mathrm{i}h} = \mathrm{e}^{-\frac{1}{2}(h,h)}\,\mathrm{e}^{-\mathrm{i}h} \\ &= \mathrm{Weyl}(-h)\,\mathrm{e}^{0} = S\,\mathrm{Weyl}(h)\,\mathrm{e}^{0}. \end{aligned} \tag{69}$$

Applying this to the wedge situation, we obtain the Bisognano–Wichmann theorem

$$\begin{aligned} &S\,\mathrm{Weyl}(h)\,\mathrm{e}^{0} = \mathrm{e}^{\mathfrak{j}_W}\,\mathrm{e}^{\delta_W^{\frac{1}{2}}}\,\mathrm{Weyl}(h)\,\mathrm{e}^{0} \qquad h \in H_R(W) \\ &\curvearrowright S A\Omega = A^*\Omega \qquad A \in \mathcal{A}(W) \equiv \mathrm{alg}\{\mathrm{Weyl}(h) \mid h \in H_R(W)\}. \end{aligned} \tag{70}$$

The geometrical content of the theorem, namely that $\Delta^{\mathrm{i}t}$ is the Lorentz boost $U(\Lambda_W(2\pi t))$ and J is the W-associated anti-unitary reflection, is now a result of the functorial nature of the map and the geometric modular properties of the Wigner theory. The map can be applied to other causally complete convex regions than wedges, but the only geometric aspect in that case is Haag duality,

$$\begin{aligned} &\mathcal{A}(\mathcal{O}) \equiv \bigcap_{W \supset \mathcal{O}} \mathcal{A}(W) \\ &\mathcal{A}(W') = \mathcal{A}(W)' \curvearrowright \mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})'. \end{aligned} \tag{71}$$

After having used the rather compact coherent state formalism for explaining the functorial properties of the map Γ, we may return to the more common way of writing the Weyl generators in terms of creation/annihilation operators used at the beginning of this section.

An important thermal aspect of the Tomita–Takesaki modular theory is the validity of the Kubo–Martin–Schwinger (KMS) boundary condition in the state $\omega(\cdot) \equiv (\Omega, \cdot\Omega)$ with $(\mathcal{A}, \Omega)$ being a standard pair [1],

$$\omega(\sigma_{t-i}(A)B) = \omega(B\sigma_t(A)) \qquad A, B \in \mathcal{A} \tag{72}$$

i.e. the existence of an analytic function $F(z) \equiv \omega(\sigma_z(A)B)$ holomorphic in the strip $-1 < \mathrm{Im}\, z < 0$ and continuous on the boundary with $F(t - \mathrm{i}) = \omega(B\sigma_t(A))$ or briefly (72)[17].

[17] Inversely, a KMS state on a C^*-algebra leads via GNS construction to a standard pair $(\mathcal{A}, \Omega)$.

For the Weyl algebras $\mathcal{A}(W) = \mathcal{A}(H_R(W))$ this property can be easily tested on the Weyl generators by using the relation of the vacuum expectation value to the Wigner inner product

$$\langle \mathrm{Weyl}(\psi)\rangle_0 = \mathrm{e}^{-\frac{1}{2}\langle\psi\psi\rangle_0} \qquad \langle\psi,\psi\rangle_0 \equiv (\psi,\psi)_{\mathrm{Wig}} \tag{73}$$

and then checking the resulting relation involving the action of $\delta^{\mathrm{i}t}$ on the $H_R(W)$ Wigner functions. This remark links up with that made on the thermal aspect of the Wigner theory (24) and the relation to the Hawking–Unruh aspects of the (Rindler) wedge localization of quantum matter. The validity of a geometric form for the Tomita–Takesaki modular theory (with J equal to the TCP operator times a spatial rotation and $\Delta^{\mathrm{i}t} = U(\Lambda_W(2\pi t))$) for general interacting wedge-localized operator algebras of QFT has been shown by Bisognano and Wichmann [1] under the assumption that the operator algebras possess point-like field generators. Recently, Mund gave a direct algebraic proof based on the use of scattering theory [42].

For half-integer spin, the Weyl functor has to be replaced by the Clifford functor R. In the previous section, we have already noted that there exists a mismatch between the geometric and the spatial complement which led to the incorporation of an additional phase factor i into the definition of j.

A Clifford algebra is associated with a real Hilbert space H_r with generators

$$R : H_r \to B(H_r) \qquad f \to R(f) \in B(H_r) \tag{74}$$

$$\begin{aligned} &R^2(f) = (f,f)_r 1 \quad \text{or} \quad \{R(f), R(g)\} = 2(f,g)_r \\ &(f,g)_r = \mathrm{Re}(f,g) \end{aligned} \tag{75}$$

where the last line is meant to indicate that we simply create a real space from a complex one by stipulating the inner product to be the real part of complex H. This links up nicely with our previous observation that for half-integer spin one should change from the symplectic structure (which goes with the imaginary part of the Wigner inner product) to the real orthogonal structure which absorbs the factor i which arises in the transition from the symplectic to the geometric complement. The fact that now the geometric complement is defined with respect to the same inner product as that of the Hilbert space makes the operator formalism in some aspects simpler. In particular, the operators $R(f)$ which are real-linear are bounded and do not need a Weyl kind of exponentiation.

These $R(f)$ generate $\mathrm{Cliff}(H_r)$ as polynomials of R. The norm is uniquely fixed by the algebraic relation, e.g.

$$\|R(f)\|^2 = \|R(f)^*R(f)\| = \|R^2(f)\| = \|f\|_r^2 \tag{76}$$

and similarly for all polynomials, i.e. on all $\mathrm{Cliff}(H_r)$. The norm closure of the Clifford algebra is sometimes called $CAR(H_r)$ (canonical anticommutation) C^*-algebra. It is unique (always up to C^*-isomorphisms) and has no ideals. This Clifford map may be used as the analogue of the Weyl functor in the case of half-integer spin $s = \frac{n}{2}$, n odd.

In using this R-formalism as a functor in order to convert the Wigner space H_{Wig} wavefunctions into Hermitian operators, one must use Schwinger's doubling formalism for the $SU(2)$ rotation group in order to have real Wigner rotations because one wants the real inner product to be invariant under an orthogonal (i.e. unitarity adjusted to reality) representation of the rotation. This is achieved by the Schwinger doubling: if ψ is a multi-component wavefunction which transforms with an $SU(2)$ matrix D, then $\psi + \psi^*$ and $\frac{1}{\mathrm{i}}(\psi - \psi^*)$ form a system which transforms with a orthogonal matrix according to

$$\begin{pmatrix} \mathrm{Re}\,D & \mathrm{Im}\,D \\ -\mathrm{Im}\,D & \mathrm{Re}\,D \end{pmatrix}. \tag{77}$$

The connection with the more common notation in terms of particle/antiparticle creation and annihilation operators is given by the following formula (and its Hermitian adjoint) with the expected anticommutation relations,

$$\begin{aligned} C^*(\psi) &= \int (a^*(p), (D^{(s)}(\mathrm{i}\sigma_2)b)(p)) \begin{pmatrix} \psi^a(p) \\ \psi^b(p) \end{pmatrix} \frac{\mathrm{d}^3p}{2\omega} \\ \{C(\varphi)C^*(\psi)\} &= (\varphi, \psi) \equiv \int \overline{(\varphi^a(p), \varphi^b(p))} \begin{pmatrix} \psi^a(p) \\ \psi^b(p) \end{pmatrix} \frac{\mathrm{d}^3p}{2\omega} \\ a(p)\Omega &= 0 \qquad b(p)\Omega = 0 \end{aligned} \tag{78}$$

where we have combined particle creation and antiparticle annihilation using the same vector notation as in (63). The presence of the matrix $D^{(s)}$ $(\mathrm{i}\sigma_2)$ assures that particle and antiparticle components have the same transformation properties. The relation with the previous Clifford algebra formalism is given by

$$\begin{aligned} R(\psi) &= \frac{1}{\sqrt{2}}(C(\psi) + C^*(\psi)) \\ R(\mathrm{i}\psi) &= \frac{\mathrm{i}}{\sqrt{2}}(C(\psi)) - C^*(\psi) \end{aligned} \tag{79}$$

where, as explained before, $\psi, \mathrm{i}\psi$ are considered as two different vectors of a real Hilbert space H_r; the associated real orthogonal inner product results automatically by computing the anticommutation relations of the R. Note that in this form the analogy with (63) is obvious. The operators which generate the localized algebras $\mathcal{A}(\mathcal{O})$ are in both cases obtained by restricting the Hilbert space of wavefunctions to the respective real modular subspaces. This restriction brings about the relation between ψ^a and $\overline{\psi^b}$ at opposite mass shell values (opposite sides of the analytic strip). The vanishing of the anticommutator instead of the commutator comes from the geometric twist factor which appears for half-integer spin. The change from symplectic to orthogonal complement is equivalent to the change of commutator to anticommutator within the two-point function. Since the higher point functions are products of two-point function, the operator anticommutation follows.

The computational rules for polynomials of $C^\#$ applied to the vacuum are simple; antisymmetrized tensor product n-particle vectors are obtained by n-fold application of $C^\#(\psi_i)$ to Ω where ψ_i runs through a basis system in the one-particle space. The CAR functor Γ encodes the action of one-particle operators in H_{Wig} into a tensor product action on the tensor product spaces. The result is again that the relation in Wigner space $\mathfrak{s} = \mathfrak{j}\delta^{\frac{1}{2}}$ for the wedge region passes to the Bisognano–Wichmann relation $S = J\Delta^{\frac{1}{2}}$ with $\Delta^{\mathrm{i}t}$ being the wedge preserving Lorentz boost in the Fock space and the W-associated involution J which differs however from the geometric reflection J_{geo} by a twist operator T,

$$\begin{aligned} &SA\Omega = A^*\Omega \qquad A \in \mathrm{alg}\{C^\#(\psi) | \psi \in H_R(W)\} \\ &S = J\Delta^{\frac{1}{2}} \qquad J = TJ_{\mathrm{geo}} \qquad T = \frac{1 - \mathrm{i}U(2\pi)}{1 - \mathrm{i}} = \begin{cases} 1 & \text{on even} \\ \mathrm{i} & \text{on odd.} \end{cases} \end{aligned} \tag{80}$$

The twist operator T is nothing but the Fock space version of the twist factor i in the Wigner theory, i.e. $T = \Gamma\mathrm{i}$. The algebra of the geometric complement is given as

$$\begin{aligned} &\mathrm{Ad}\, J_{\mathcal{O},\mathrm{geo}}\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}') \\ &\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})^t \qquad \mathcal{A}(\mathcal{O})^t \equiv \mathrm{Ad}\, T\mathcal{A}(\mathcal{O})'. \end{aligned} \tag{81}$$

Here the second line expresses the ‘twisted’ Haag duality. We leave the easy verification of some of the details to the reader.

The bosonic CCR (Weyl) and the fermionic CAR (Clifford) local operator algebras are the only ones which permit a functorial interpretation in terms of a 'quantization' of classical function algebras. In the next section, we will explain why these operator algebras are the only QFTs which possess sub-wedge-localized 'PFG' one-particle creation operators.

It is a remarkable fact that the operator algebras associated with the $m = 0, \kappa \neq 0$ Wigner representation are obtained as in the above cases by applying the CCR/CAR functor to the Wigner space; the only difference being that for $\kappa \neq 0$ there are no generating point-like fields, in fact there are no operators with a localization region which is below space-like cones. It is a plausible conjecture that the point-like covariant fields in this case are to be replaced by covariant semi-infinite string-like generators[18] inasmuch as point-like fields are limiting cases of double cone localization. In both cases, the idealizations are the result of the absence of any elementary length which requires a smallest diameter of the double cone (respectively a minimal opening angle of a space-like cone). 'Stringy' objects are further removed from classical geometric constructions and therefore less (perhaps not at all) accessible by quantization approaches. Whereas a covariant tensor/spinor calculus for point-like objects was already available before the Wigner representation theory, there is no ready-made classical (Lagrangian, equation of motion) formalism for strings which could obviate the localization aspects in an analogous way as the covariant point-like fields do. The presence of string localization in the Wigner classification is somewhat of a surprise. The present modular approach shows that the localization properties can be understood in terms of space-like cones even before the idealized limits which shrink these operators to operator-valued string distributions have been performed. In fact, the present approach may even help in finding them. The fact that in pursuing a seemingly different problem one stumbles upon a *natural string field theory* (the 'natural' refers to their origin from the principles underlying Wigner's positive energy representation classification program) is really very interesting especially since mathematically controllable string field theories are very rare. The reason why these Wigner strings were overlooked is probably that they lack a Lagrangian quantization formulation. Another reason may be sociological since by declaring that nature has no use for them [8] before string theory became popular, they probably got lost in the collective knowledge.

In the case of $d = 1 + 2$ anyonic spin representations, the presence of a twist factor has a more radical consequence. Whereas the fermionic twist is still compatible with the existence of PFGs and free fields in Fock space, the twist associated with genuine braid group statistics causes the presence of vacuum polarization for any sub-wedge-localization region.

In the general case of an interacting theory in $d = 1 + 3$ with compact localization (which according to the DHR analysis is necessarily a theory of interacting bosons/fermions), the modular setting for the wedge algebras is modified by the presence of the scattering operator

$$\begin{aligned} &S = J\Delta^{\frac{1}{2}} \\ &\Delta_W^{\mathrm{i}t} = U(\Lambda_W(2\pi t)) \qquad J_W = J_{W,0} S_{\mathrm{scat}}. \end{aligned} \tag{82}$$

The interaction enters through a modification of the modular involution by the scattering matrix S_{scat}. This formula is derived on the basis of the validity of scattering theory; a sufficient condition is the presence of a mass gap which spectrally separates the one-particle states and the validity of asymptotic completeness. The modular unitary $\Delta^{\mathrm{i}t}$ is unaffected (as are all Poincaré transformations from $\tilde{P}_+^\uparrow$). The last line which expresses the change of the modular involution from its free value in the free incoming theory to the actual J

[18] Note that point- or string-like limits are only meaningful in the Fock space formulation, i.e. after the application of the CCR/CAR functor, but not in the Wigner wavefunction space.

is nothing but another way of writing the TCP invariance of the scattering operator (using $J = \mathrm{TCP} \cdot \mathrm{rot}_W(\pi)$) [43]. The above formula gives S_{scat} the status of a relative modular invariant (between the interacting and the incoming free wedge algebra).

Knowing the operators which appear in these modular properties of wedge algebras, but lacking direct information about the wedge algebra itself, the only thing one can do is to study the change of modular subspaces by solving modular equations for $\mathcal{H}_R(W) = \{\psi | S\psi = \psi\}$. Necessary and sufficient conditions for such a spatial modular theory to originate from an operator theory have been elaborated by Connes [44]. They use the so-called natural modular cones $\mathcal{P}_{\mathcal{A}(W),\Omega} = \{AJAJ\Omega | A \in \mathcal{A}(W)\}$ and assume rather detailed properties about their facial substructure. It is presently unknown whether these conditions have a physical implementation. It is comforting to know that even though the modular setting based on (82) does not lead to the actual construction of an AQFT, it suffices to show that if there is any local net of QFT algebras behind an admissible S-matrix (unitary, crossing and analyticity), i.e. $S_{\mathrm{scat}} \to \{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{K}}$, it is unique [14]. This uniqueness argument uses besides the modular structure of vectors of the form $A\psi$ with $A \in \mathcal{A}(\mathcal{O}) \subset \mathcal{A}(W)$ and $\psi \in H(W)$ (for modular restricted n-particle incoming vectors) the full crossing symmetry of form factor spaces. It underlies the calculation of bootstrap form factor program for factorizable models. Despite the fact that crossing of incoming particles from bra-vectors to analytically continued outgoing antiparticles in matrix elements of local operators belongs to a 50 year old folklore of QFT in the Lehmann–Symanzik–Zimmermann setting, it has not been derived from the general principles in the generality needed here. Nevertheless a unicity proof based on its use is not without interest especially in view of the fact that attempts to derive this from the principles of Wightman QFT remained without success even for $S = 1$ [45]. Although the domain properties of the Tomita operator $\mathfrak{s}$ on Wigner wavefunctions show that this operator transforms particle in analytically extended antiparticle states, a modular understanding of crossing in the presence of interactions remains still a problem for the future.

The modular based approach which tries to use the twist/S-matrix factor in $J = J_0 T$, respectively $J = J_0 S_{\mathrm{scat}}$, for the determination of the algebraic structure of $\mathcal{A}(W)$ and subsequently computes the net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{K}}$ by forming intersections is presently limited to theories which permit only virtual but no real particle creation. Besides the exceptional Wigner representation (anyons, spin towers) which lead to a twist and changed space-like commutation relations, the only standard (bosonic, fermionic) interacting theories are the $S_{\mathrm{scat}} = S_{\mathrm{el}}$ models of the $d = 1 + 1$ bootstrap form factor setting (factorizing models). In that case, there exist unbounded operators affiliated with the wedge algebra with nice mathematical properties; if applied once to the vacuum, they create a one-particle vector without admixture of particle/antiparticle vacuum polarization clouds. For these reasons, they are called 'tempered PFGs' where the PFG refers to their *polarizarion free generation* of one-particle states and the tempered refers to their well-behaved Fourier transform [46]. For those cases, the algebraic construction program which starts with wedge algebras links up with the bootstrap form factor approach [19, 48]; in fact, it provides a spacetime interpretation of the Zamolodchikov–Faddeev algebra and explains the various recipes in terms of the general principles of local quantum physics. For models outside this special class, particularly higher dimensional ones, there exist very promising constructive ideas which are however more involved and still in their initial state of development: the algebraic lightfront or algebraic holography approach. Here the starting idea is that the wedge algebra is equal to the algebra on its lightfront horizon but there spacetime net structures are quite different. The net structure on the lightfront is that of a chiral conformal algebra with only canonical scale dimensions in light ray direction and a quantum mechanical (absence of vacuum polarization) behaviour in transverse direction. Some more remarks can be found at the end of section 5.

For those readers who are familiar with the textbook method [8] of passing from Wigner representation to covariant point-like free fields, it may be helpful to add a remark which shows the connection to the modular approach. For writing covariant free fields in the (m, s) Fock space

$$\psi^{[A,\dot{B}]}(x) = \frac{1}{(2\pi)^{3/2}} \int \left\{ e^{-ipx} \sum_{s_3} u(p_1, s_3) a(p_1, s_3) + e^{ipx} \sum_{s_s} v(p_1, s_3) b^*(p_1, s_3) \right\} \frac{d^3 p}{2\omega} \tag{83}$$

where $a^\#, b^\#$ are creation/annihilation operators of Wigner (m, s) particles and $\psi^{[A,\dot{B}]}$ are covariant dotted/undotted fields in the $SL(2, C)$ spinor formalism, it is only necessary to find intertwiners

$$u(p) D^{(s)}(\tilde{R}(\tilde{\Lambda}, p)) = D^{[A,\dot{B}]}(\tilde{\Lambda}) u(\tilde{\Lambda}^{-1} p) \tag{84}$$

between the Wigner $D^{(s)}(\tilde{R}(\tilde{\Lambda}, p))$ and the covariant $D^{[A,\dot{B}]}(\tilde{\Lambda})$ and these exist for all $A, \dot{B}$ which relative to the given s obey

$$|A - \dot{B}| \leqslant s \leqslant A + \dot{B}. \tag{85}$$

For each of these infinitely many values $(A, \dot{B})$ there exists a rectangular $(2A + 1)(2\dot{B} + 1) \times (2s + 1)$ intertwining matrix $u(p)$. Its explicit construction using Clebsch–Gordan methods can be found in Weinberg's book [8]. Analogously, there exist antiparticle (opposite charge) intertwiners $v(p)$: $D^{(s)*}(R(\Lambda, p) \longrightarrow D^{[A,\dot{B}]}(\Lambda)$. All of these mathematically different fields in the same Fock space describe the same physical reality; they are just the linear part of a huge local equivalence class and they do not exhaust the full 'Borchers class' which consists of all Wick-ordered polynomials of the $\psi^{[A,\dot{B}]}$. They generate the same net of local operator algebras and in turn furnish the singular coordinatizations. Free fields for which the full content of formula (83) can be described by the totality of all solutions of an Euler–Lagrange equation exist for each (m, s) but are very rare (for example, Rarita–Schwinger for $s = \frac{3}{2}$). It is a misconception that they are needed for physical reasons. The causal perturbation theory can be done in any of those field coordinates and that one needs Euler–Lagrange fields in the setting of Euclidean functional integrals is an indication that differential geometric requirements and quantum physical ones do not always go in the same direction.

An important difference to the field coordinatization free approach is the fact that the latter does not come with any notion of short-distance behaviour which is typical for the kind of chosen field coordinate. Neither in the standard cases of localization in arbitrary small double cones, nor in the exceptional cases where there is no better than space-like cone localization does the algebraic formulation reveal anything about the short-distance behaviour of field or string coordinates. In fact, the role of point-like or string-like fields is akin to the use of singular coordinates in differential geometry. Physical invariants such as the S-matrix, which are only dependent on local equivalence classes of fields, are the natural analogues of geometric invariants. However there is one black cloud in this analogy; a closer look reveals that the point-like fields are behaving like singular coordinates which cannot be trusted if it comes to defining the intrinsic frontiers between well-defined and nonsensical QFT. In particular, it is not clear whether non renormalizable in the standard power counting of perturbation theory is the same as physically invalid or mathematically inconsistent. The existence of a bootstrap-formfactor construction without ultraviolet divergencies for a particular family of $d = 1 + 1$ models in which wedge localization plays a prominent role (59) nourishes some hope for the existence of a more general on-shell approach which generalizes Wigner's representation theoretic spirit with the help of modular ideas about operator algebras to the realm of interactions.

An illustration that shows ambiguities in the standard approach to interacting higher spin particles is provided by the perturbation theory involving massive vector mesons. Since the operator dimensions of a physical (m, $s = 1$) field-coordination in the short distance sense cannot be better than dim $A_\mu = 2$ (and therefore by one unit larger than its classically value in a gauge theoretical setting), any interaction polynomial (which must be at least of degree 3) will have a short-distance dimension of at least 5 and therefore be classified as nonrenormalizable. However, there is a well-known BRST trick to overcome this barrier which in the present Wigner representation setting consists in finding a cohomological representation in which the physical (m, $s = 1$) representation emerges as the result of 'modding out' the range of a δ-operator which acts in a larger space. It turns out that the unphysical extended Wigner representation allows for a coordinatization in terms of dim = 1 unphysical fields. The cohomological nature of the representation guarantees that one can descend back to physics by undoing the cohomological trick after the calculation. By using this unphysical 'catalyser' one finally obtains a physical vector meson field A_μ whose short-distance behaviour is only logarithmic corrections away from its free field value; in addition one discovers the presence of a new physical degree of freedom which was not put in at the start. This phenomenon is mostly subsumed under the heading Higgs mechanism in gauge theory, but whereas this may have been a useful point of view for becoming aware of how to get vector mesons into the renormalization setting, it is less helpful for a deeper physical explanation. In fact, the observation that there exists precisely one renormalizable interaction between vector mesons and with lower spin matter (and that the existence of Higgs degrees of freedom is a consequence of this uniqueness) asks for a more profound explanation outside the present framework. Since in the case of a unique interaction no selection principle is required, the necessity of a gauge principle exists only in the classical theory where a vector potential can be coupled in several ways. In other words, the ill-understood but clear unicity observation should be taken as a tentative local quantum physical explanation for the (semi)classical gauge principle rather than the other way around. This would imply the admission that even on a perturbative level the present ideas on good and bad short-distance behaviour are open to doubts and the question of whether the frontier is drawn correctly by Lagrangian quantization (leading invariably to singular point-like field coordinatization) is wide open. Further doubts about the relevance of the short-distance classification obtained by power counting in the Lagrangian quantization approach come from the bootstrap form factor constructions of $s = 1 + 1$ factorizable models (see section 5). This construction deals with form factors and avoids the integration over intermediate off-shell momenta which are the cause of ultraviolet divergencies. All fields of the local equivalence class are treated democratically and the only hierarchical structure of particles is that of charges and their fusion. There are no short-distance impositions and all fields in the equivalence classes have finite short-distance scaling powers.

Some of these problems have an interesting history. In the 1960s after the first excitement about the success of renormalization subsided, there was a strong desire to do particle physics without any field coordinatization by directly studying the most important invariant of local equivalence classes of fields, namely the S-matrix. Restrictive properties besides unitarity which keep certain properties 'as if' an S-matrix would have resulted from an underlying QFT were formulated in the form of crossing properties and their analytic prerequisites. But the lack of bringing crossing into an operative form eventually led to a credibility loss in this approach. In fact, to date not even a perturbative way of handling on mass shell unitarity and crossing has been found apart from the successful bootstrap form factor program for factorizable models in $d = 1 + 1$ dimensions (see below). The present attempt to deal with interactions in a similar intrinsic spirit as Wigner did in the absence of interactions may be seen as a partial revival of these old dreams of finding formulations which steer clear of ultraviolet problems, but now with the backing of the powerful operator algebra approach, in particular the modular localization theory, and certainly without the old ideology of cleansing the formulation of all off-shell concepts.

4. Vacuum polarization and breakdown of functorial relations

The functorial relations of the previous section between Wigner subspaces and operator algebras are strictly limited to the standard half-integer spin representations including the Wigner helicity towers. Plektons (in particular the Wigner $d = 1 + 2\ s \neq$ half-integer anyons), i.e. operators generating particles with braid group statistics and interacting particles, do not permit a direct functorial relation between wavefunction spaces and operator algebras.

In order to understand the physical mechanism which prevents such a functorial relation, it is instructive to look directly the operator algebras. Given an operator algebra $\mathcal{A}(\mathcal{O})$ localized in a causally closed region $\mathcal{O}$ with a nontrivial causal complement $\mathcal{O}'$ (so that $(\mathcal{A}(\mathcal{O}), \Omega)$ is standard pair) we may ask whether this algebra admits a 'polarization free generator' (PFG) [6], namely an affiliated possibly unbounded closed operator G such that Ω is in the domain of G, G^* and $G\Omega$ and $G^*\Omega$ are vectors in $E_m H$ with the E_m projector on the one-particle space associated with an isolated mass shell of mass m.

It turns out that if one admits a sufficiently crude localization such as that in wedges, one can reconcile the standardness of the pair $(\mathcal{A}(W), \Omega)$ (i.e. physically the unique $A\Omega \leftrightarrow A \in \mathcal{A}(W)$ relationship) with the absence of polarization clouds caused by localization. For the convenience of the reader, we will recall some of the theorems which relate modular theory and PFGs.

An interesting situation emerges if these PFG operators which always generate a dense one-particle subspace also generate an algebra of unbounded operators which is affiliated to a corresponding von Neumann algebra $\mathcal{A}(\mathcal{O})$. For causally complete sub-wedge regions $\mathcal{O}$ such a situation inevitably leads to interaction-free theories, i.e. the local algebras generated by ordinary free fields turn out to be the only $\mathcal{A}(\mathcal{O})$-affiliated PFG. Such a situation is achieved by domain restrictions on the (generally unbounded) PFGs which are tantamount to their temperedness in the sense of existence of Fourier transforms. Without such restriction, it would be difficult to imagine a constructive use of PFG [46].

Before studying PFG, it is helpful to remind the reader of the following theorem of general modular theory.

Theorem 1. *Let S be the modular operator of a general standard pair $(\mathcal{A}, \Omega)$ and let Φ be a vector in the domain of S. There exists a unique closed operator F affiliated with F (notation $F\eta\mathcal{A}$) which together with F^* has the reference state Ω in its domain and satisfies*

$$F\Omega = \Phi \qquad F^*\Omega = S\Phi. \tag{86}$$

A proof of this and the following theorem can be found in [46].

For the special field theoretic case $(\mathcal{A}(W), \Omega)$, the domain of S which agrees with that of $\Delta^{\frac{1}{2}} = \mathrm{e}^{\pi K}$, $K =$ boost generator has evidently a dense intersection $\mathcal{D}^{(1)} = H^{(1)} \cap \mathcal{D}_{\Delta^{\frac{1}{2}}}$ with the one-particle space $H^{(1)} = E_m H$. Hence the operator F for $\Phi^{(1)} \in \mathcal{D}^{(1)}$ is a PFG G as previously defined. However, the abstract theorem contains no information on whether the domain properties admit a repeated use of PFG similar to smeared fields in the Wightman setting, nor does it provide any clue about the position of a dom G relative to scattering states. Without such a physically motivated input, wedge-supported PFG would not be useful. An interesting situation is encountered if one requires the G to be tempered. Intuitively speaking, this means that $G(x) = U(x)GU(x)^*$ has a Fourier transform as needed if one wants to use PFG in scattering theory. If one in addition assumes that the wedge algebras to which the PFGs are affiliated are of the standard Bose/Fermi type, i.e. $\mathcal{A}(W') = \mathcal{A}(W)'$ or the twisted Fermi commutant $\mathcal{A}(W)^{tw}$, one finds

Theorem 2. *PFGs for the wedge localization region always exist, but the assumption that they are tempered leads to a purely elastic scattering matrix $S_{\text{scat}} = S_{\text{el}}$, whereas in $d > 1 + 1$ is only consistent with $S_{\text{scat}} = 1$.*

Together with the recently obtained statement about the uniqueness of the inverse problem in the modular setting of AQFT [14] one finally arrives at the interaction-free nature in the technical sense that the PFG can be described in terms of free Bose/Fermi fields.

The nonexistence of PFG in interacting theories for causally completed localization regions smaller than wedges (i.e. intersections of two or more wedges) can be proved directly, i.e. without invoking scattering theory.

Theorem 3. *PFGs localized in smaller than wedge regions are (smeared) free fields. The presence of interactions requires the presence of vacuum polarization in all state vectors created by applying operators affiliated with causally closed smaller wedge regions.*

The proof of this theorem is an extension of the ancient theorem [43] that point-like covariant fields which permit a frequency decomposition (with the negative frequency part annihilating the vacuum) and commute/anticommute for space-like distances are necessarily free fields in the standard sense. The frequency decomposition structure follows from the PFG assumption and the fact that in a given wedge one can find PFG whose localization is space-like disjoint is sufficient for the analytic part of the argument to still go through, i.e. the point-like nature in the old proof is not necessary to show that the (anti)commutator of two space-like disjoint localized PFG is a c-number (which only deviates from the Pauli–Jordan commutator by its lack of covariance). The most interesting aspect of this theorem is the inexorable relation between interactions and the presence of vacuum polarization which for the first time leads to a completely intrinsic definition of interactions which is not based on the use of Lagrangians and particular field coordinates. This poses the interesting question how the shape of the localization region (e.g. size of the double cone) and the type of interaction are related with the form of the vacuum polarization clouds which necessarily accompany a one-particle state. We will make some comments in the next section.

As Mund has recently shown, this theorem has an interesting extension to $d = 1 + 2$ QFT with braid group (anyon) statistics.

Theorem 4 ([47])**.** *There are no PFGs affiliated to field algebras localized in space-like cones with anyonic commutation relations, i.e. sub-wedge-localized fields obeying braid group commutation relations applied to the vacuum are always accompanied by vacuum polarization clouds. Even in the absence of any genuine interactions, this vacuum polarization is necessary to sustain the braid group statistics and to maintain the spin-statistics relation.*

This poses the interesting question whether quantum mechanics is compatible with a nonrelativistic limit of braid group statistics. The nonexistence of vacuum polarization-free locally (sub-wedge) generated one-particle states suggests that as long as one maintains the spin-statistics connection throughout the nonrelativistic limit procedure, the result will preserve the vacuum polarization contributions and hence one will end up with nonrelativistic field theory instead of quantum mechanics[19].

[19] The Leinaas–Myrheim geometrical arguments [49] do not take into account the true spin-statistics connection.

Using the concept of PFG, one can also formulate this limitation of quantum mechanics against the incorporation of any other commutation relations than those associated with Bose/Fermi statistics in a more provocative way by stating that (using the generally accepted dictum that QFT is more fundamental than QM) QM owes its physical relevance to the fact that the permutation group (boson/fermion) statistics permits sub-wedge-localized PFG (i.e. free fields which create one-particle states without vacuum polarization admixture) whereas the more general braid group statistics does not.

Another problem which even in the Wigner setting of noninteracting particles is interesting and has not yet been fully understood is the pre-modular theory for disconnected or topologically nontrivial regions, e.g. in the simplest case for disjoint double intervals of the massless $s = \frac{1}{2}$ chiral model on the circle. Such situations give rise to nongeometric (fuzzy) 'quantum symmetries' of purely modular origin without a classical counterpart.

5. Construction of models via modular localization

Since to date more work has been done on the modular construction of $d = 1 + 1$ factorizing models, we will first illustrate our strategy in this case and then make some comments of how we expect our approach to work in the case of higher dimensional $d = 1 + 2$ anyons.

The construction consists basically of two steps: first one classifies the possible algebraic structures of tempered wedge-localized PFG and then one computes the vacuum polarization clouds of the operators belonging to the double cone intersections.

Let us confine ourselves to the simplest model which we may associate with a massive self-conjugate scalar particle without bound states. If there is no interaction, the appropriate theorem of the previous section would only leave the free field as a PFG for wedge and any sub-wedge localization,

$$\begin{aligned} A(x) &= \frac{1}{\sqrt{2\pi}} \int (\mathrm{e}^{-\mathrm{i}p(\theta)x} a(\theta) + \mathrm{e}^{\mathrm{i}p(\theta)x} a^*(\theta))\, \mathrm{d}\theta \\ A(f) &= \int A(x) \hat{f}(x)\, \mathrm{d}^2x = \frac{1}{\sqrt{2\pi}} \int_C a(\theta) f(\theta)\, \mathrm{d}\theta \qquad \operatorname{supp} \hat{f} \in W \\ p(\theta) &= m(\cosh\theta, \sinh\theta) \qquad C : \int_{-\infty}^{+\infty} \ldots \mathrm{d}\theta + \int_{-\infty-\mathrm{i}\pi}^{+\infty-\mathrm{i}\pi} \ldots \mathrm{d}\theta \end{aligned} \tag{87}$$

where in order to put into evidence that the mass shell in two dimensions is a one-parametric manifold we have used the rapidity parametrization in which the plane wave factor is an entire function in the complex extension of θ with $p(\theta - \mathrm{i}\pi) = -p(\theta)$. The formula has been written in terms of the smeared field with the support of the test function $\hat{f}$ in the right wedge in order to motivate the notation as a contour integral over C which involves the mass shell restriction of the analytic and integrable Fourier transform (written in terms of the rapidity variable θ) at the two boundaries of the rapidity strip $-\mathrm{i}\pi < \operatorname{Im}\theta < 0$. Remembering from the previous section that tempered PFGs stay close to noninteracting operators in that only elastic scattering is permitted, we make the ansatz

$$\begin{aligned} G(x) &= \frac{1}{\sqrt{2\pi}} \int (\mathrm{e}^{-\mathrm{i}px} Z(\theta) + \mathrm{e}^{\mathrm{i}px} Z^*(\theta))\, \mathrm{d}\theta \\ G(\tilde{f}) &= \frac{1}{\sqrt{2\pi}} \int_C Z(\theta) f(\theta)\, \mathrm{d}\theta \end{aligned} \tag{88}$$

where the Z are defined on the incoming n-particle vectors by the following formula for the action of $Z^*(\theta)$ for the rapidity-ordering $\theta_{\mathrm{i}} > \theta > \theta_{i+1}, \theta_1 > \theta_2 > \cdots > \theta_n$:

$$Z^*(\theta)a^*(\theta_1)\dots a^*(\theta_i)\dots a^*(\theta_n)\Omega$$
$$= S(\theta-\theta_1)\dots S(\theta-\theta_i)a^*(\theta_1)\dots a^*(\theta_i)a^*(\theta)\dots a^*(\theta_n)\Omega$$
$$+ \text{contributions from bound states.} \tag{89}$$

Although the coefficient function S is the two-particle S-matrix, this interpretation does not have to be imposed; it will be a consequence of (82) as soon as we establish that the $Z^\#$ are the creation/annihilation operators of a wedge-localized algebra. In the absence of bound states (which we assume in the following) this amounts to the commutation relations[20]

$$\begin{aligned} Z^*(\theta)Z^*(\theta') &= S(\theta-\theta')Z^*(\theta')Z^*(\theta) \qquad \theta<\theta' \\ Z(\theta)Z^*(\theta') &= S(\theta'-\theta)Z^*(\theta')Z(\theta)+\delta(\theta-\theta') \end{aligned} \tag{90}$$

where the structure functions S must be unitary in order that the Z-algebra be a *-algebra. It is easy to show that as a result of its proximity to free creation/annihilation operators the domain of the Z is identical to that of the free theory. We still have to show that our 'nonlocal' G are wedge-local. According to modular theory for this we have to show the validity of the KMS condition. It is very gratifying that the KMS condition for operators $G(\hat{f})$ with supp $\hat{f}\subset W$ which are affiliated to the algebra $\mathcal{A}(W)$ is equivalent to the crossing property of the S.

Proposition 5 [6, 17]**.** *The PFGs with the above algebraic structure for the Z are wedge-localized if and only if the structure coefficients $S(\theta)$ in (90) are meromorphic functions which fulfil crossing symmetry in the physical θ-strip, i.e. the requirement of wedge localization converts the Z-algebra into a Zamolodchikov–Faddeev algebra.*

Improving the support of the wedge-localized test function in $G(\hat{f})$ by choosing the support of $\hat{f}$ in a double cone well inside the wedge does not improve loc $G(\hat{f})$; it is still spread over the entire wedge. This is certainly very different from the behaviour of smeared point-like fields.

By forming an intersection of two oppositely oriented wedge algebras, one can compute the double cone algebra or rather (since the control of operator domains has not yet been accomplished) the spaces of double cone localized bilinear forms (form factors of would-be operators).

The most general operator A in $\mathcal{A}(W)$ is an LSZ-type power series in the Wick-ordered Z,

$$A=\sum\frac{1}{n!}\int_C\dots\int_C a_n(\theta_1\dots\theta_n):Z(\theta_1)\dots Z(\theta_n):\mathrm{d}\theta_1\dots\mathrm{d}\theta_n \tag{91}$$

$$A\in\mathcal{A}_{\text{bil}}(W) \tag{92}$$

with strip-analytic coefficient functions a_n which are related to the matrix elements of A between incoming ket and outgoing bra multiparticle state vectors (form factors). The integration path C consists of the real axis, associated with annihilation operators, and the line $\mathrm{Im}\,\theta=-\mathrm{i}\pi$, corresponding to creators. Writing such power series without paying attention to domains of operators means that we are only dealing with these objects (as in the LSZ formalism) as bilinear forms (92) or form factors whose operator status still has to be settled.

Now we come to the second step of our algebraic construction: the computation of double cone algebras. The spaces of bilinear forms which have their localization in double cones

[20] In the presence of bound states, such commutation relations only hold after applying suitable projection operators.

are characterized by their relative commuting (with an obvious change for fermions or more general statistics) with shifted generators $A^{(a)}(f) \equiv U(a)A(f)U^{*}(a)$

$$[A, A^{(a)}(f)] = 0 \qquad \forall f \operatorname{supp} f \subset W \qquad A \subset \mathcal{A}_{\mathrm{bil}}(C_a) \tag{93}$$

where the subscript indicates that we are dealing with spaces of bilinear forms (form factors of would-be operators localized in C_a) and not yet with unbounded operators and their affiliated von Neumann algebras. This relative commutant relation [6] on the level of bilinear forms is nothing but the famous 'kinematical pole relations' which relate the even a_n to the residuum of a certain pole in the a_{n+2} meromorphic functions. The structure of these equations is the same as that for the form factors of point-like fields; but whereas the latter lead (after splitting off common factors [19] which are independent of the chosen field in the same superselection sector) to polynomial expressions with a hard-to-control asymptotic behaviour, the a_n of the double cone localized bilinear forms are solutions which have better asymptotic behaviour controlled by the Paley–Wiener–Schwartz theorem. We will not discuss here the problem of how this improvement can be used in order to convert the bilinear forms into genuine operators. Although we think that this is largely a technical problem which does not require new concepts, the operator control of the second step is of course important in order to convince our constructivist friends that modular methods really do provide a rich family of nontrivial $d = 1 + 1$ models. We hope to be able to say more in future work.

The extension to the general factorizing $d = 1 + 1$ models should be obvious. One introduces multi-component $Z^{\#}$ with matrix-valued structure functions S. The contour deformation from the original integral to the 'crossed' contour which is necessary to establish the KMS conditions in the presence of bound state poles in the physical θ-strip compensates those pole contributions against the bound state contributions in the state vector ansatz (89) [6].

As a side remark, we add that the $Z^{\#}$ operators are conceptually somewhere between the free incoming and the interacting Heisenberg operators in the following sense: whereas any particle state in the theory contributes to the structure of the Fock space and has its own incoming creation/annihilation operator, the $Z^{\#}$ operators are (despite the rather rough wedge localization properties of their spacetime related PFG G) similar to charge-carrying local Heisenberg operators in the sense that all other operators belonging to particles whose charge is obtained by fusing that of Z and Z^{*} are functions of Z analogous to the bound state fusion of Heisenberg operators [50]. In other words, the particle–field duality which holds for free fields already becomes invalidated by the interacting wedge-localized PFG operators G before one gets to the double cone localized operators which constitute the algebraic substitute of the point-like Heisenberg operators.

There are good indications that the present method, which starts from wedge-localized tempered PFG and obtains the smaller algebras by intersections, can also be used for the construction of the operator algebras associated with 'free' $d = 1 + 2$ Wigner anyons where the use of 'free' is meant in the sense of no additional interactions, i.e. the 'freest' possible realization of braid group statistics [47].

The impossibility of a compact localization in the case of the exceptional Wigner representation places them out of reach by Lagrangian quantization methods. The charge-carrying PFG operators corresponding to the wedge-localized subspaces as well as their best localized intersections are more 'noncommutative' than those for standard QFT and the worsening of the best possible localization is inexorably interwoven with the increasing space-like noncommutativity. This kind of noncommutativity should however be kept apart from the noncommutativity of spacetime itself, whose consistency with the Wigner representation theory will be briefly mentioned in the last section.

This leaves the question of how to go about getting a constructive hold on the structure of wedge algebras for QFT outside these special families of factorizing models and free anyons. In this context a recent method which focuses on the lightfront boundary (which plays the role of a causal horizon) of the wedge looks very promising. As a quantum counterpart of the classical characteristic initial value problem, one finds that the lightfront horizon algebra is identical to that of the wedge,

$$\mathcal{A}(W) = \mathcal{A}(LFH_W) \tag{94}$$

but their local net structures are very different [28]. Only regions which have a semi-infinite extension on LFH_W into the direction of the characteristic light ray cast a causal shadow into W; all other regions are related in a 'fuzzy' way, i.e. the associated operator algebras which have a geometrical position in one description are associated with a 'spread out' subalgebra in the other description. For obvious reasons, this relation between a d-dimensional ordinary QFT and a $d-1$ 'exotic' one (the lightfront is not on a global hyperbolic space, there are no causal shadows for compact regions, no Cauchy propagation etc) is called *algebraic holography*. In fact, it implements many of 't Hooft's ideas [51] about holography except that it is not tied to curved spacetime and that in the presence of interactions there is no direct relation between the original point-like field generators and those which describe the holographic projection. The usefulness of this method lies in the enormous simplification; the projected degrees of freedom on the lightfront split into longitudinal (along the light ray direction) chiral and transverse quantum mechanical degrees of freedom [28] where the vacuum polarization structure is in the longitudinal direction. The generating fields $A(x_+, x_\perp)$ of those algebras have a commutation structure which in the simplest case is of the form

$$\begin{aligned} [A_{LF}(x_+, x_\perp), A_{LF}(x'_+, x'_\perp)] &= B_{LF}(x_+, x_\perp)\delta(x_+ - x'_+)\delta(x_\perp - x'_\perp) \\ [A_{\mathrm{chir}}(x_+), A_{\mathrm{chir}}(x'_+)] &= B_{\mathrm{chir}}(x_+)\delta(x_+ - x'_+) \end{aligned} \tag{95}$$

where $A_{\mathrm{chir}}(x_+)$ is an $A_{LF}(x_+, x_\perp)$-affiliated chiral field (i.e. the lightfront field appears as if it is the product of a chiral field with a Schrödinger field) and the general case is that of a W-algebra (or Lie-field theory) where one has a system of generating fields $A_{LF}^{(i)}(x_+, x_\perp)$ $i = 1, 2, \ldots$ and the right-hand side contains one transverse $\delta(x_\perp - x'_\perp)$ function multiplied with a finite sum of chiral δ-functions and their derivatives multiplied with dimension-matching $A_{LF}^{(i)}(x_+, x_\perp)$. For those who are familiar with the commutation relations of the chiral energy–momentum tensor and the W-algebra generalization, it suffices to say that apart from the rather trivial dependence on $x_\perp$ the commutation structure of the point-like generators of the lightfront algebras is just like that of W-algebras. This means in particular that the dimension of the affiliated chiral fields is (half-)integer which implies an enormous simplification as compared with the original algebra in the ambient space which in general has short-distance anomalous dimensions.

This structure harmonizes nicely with the fact that the 'translations' of the Wigner little group of the light ray direction become quantum mechanical transverse Galilei transformations with the light ray coordinate playing the role of the time. They do change the wedge because they transform its edge into another edge within the same lightfront. The holographic projection supplies at least a simple start, but most of the steps of recreating the d-dimensional ambient theory by morphisms on the lightfront net have not yet been elaborated.

In passing, we mention that the transversal structure which expresses the total transverse decoupling of degrees of freedom is the reason behind the area proportionality of a suitably defined entropy which measures the entanglement of the vacuum with respect to a split tensor product vacuum with a split along the edge which separates the lightfront into two halves [28]. It is quite interesting to note that the quantum version of Bekenstein's black hole area law has a (Rindler–Unruh) counterpart in Minkowski space QFT.

6. Outlook

In the past, the power of Wigner's representation theory has been somewhat underestimated. As a completely intrinsic relativistic quantum theory which stands on its own feet (i.e. does not depend on any classical quantization parallelism and thus gives quantum theory its deserved dominating position) it was used in order to back up the Lagrangian quantization procedure [8], but thanks to its modular localization structure it is capable of doing much more and also sheds new light on problems which remained outside Lagrangian quantization and perturbation theory. This includes problems where, contrary to free fields, no PFG operator (i.e. one which creates a pure one-particle state without a vacuum polarization admixture) for sub-wedge regions exists, but where wedge-localized algebras still have tempered generators as $d = 1 + 1$ factorizing models and the expected behaviour of $d = 1 + 2$ 'free' anyons.

A quite interesting observation which merits a more detailed study is that the Wigner helicity tower strings are really objects of a 'string field theory' which in fact look mathematically quite accessible since different from the anyonic $d = 1 + 2$ strings, they can be created without associated vacuum polarization clouds. This simplicity and the fact that they are 'natural' (they appear in Wigner's classification according to well-established particle physics principles and not as a result of looking for string-like objects) make them interesting objects for further research. The observation that string-like localization was implicit in Wigner's 1939 work as one of the two possible forms of best possible localizations (point-like and string-like) and that Wigner's massless helicity towers were rejected in textbooks as 'unnatural' whereas string theory was presented as the wave of the future, does not lack natural irony.

Since conformal theories in any dimensions, i.e. even beyond chiral theories, are 'almost free' (in the sense that the only structure which distinguishes them from free massless theories is the spectrum of anomalous dimension which in the algebraic approach appears to be related to an algebraic braid-like structure in the time-like direction [52]), we believe that they can also be classified and constructed by modular methods.

Another insufficiently understood problem is the physical significance of the infinitely many modular symmetry groups (with the Poincaré or conformal subgroups being the maximal vacuum-preserving diffeomorphisms) which act in a fuzzy way within the localization regions and in their causal complements [53]. An educated guess would be that they are related to the nature of the vacuum polarization clouds which local operators in that region generate from the vacuum.

The reader must not have failed to note that in the present work the name Wigner stands for more than the classification of irreducible positive energy representation of the Poincaré group. It is used in a programmatic spirit to attract attention to a field coordinatization independent approach to local quantum physics which tries to combine local quantum physics in the spirit of Haag's book [1] with some of the ideas of the S-matrix bootstrap of the 1960s which aimed at a scattering theory of Wigner particles without the intervention of fields. Before the arrival of modular theory of operator algebras, the two ideas appeared rather antagonistic[21]. But through modular theory, in particular that of wedge algebras in relation to the vacuum state, a semi-local aspect of the S-matrix emerged which is totally characteristic for S-matrices in local quantum physics. This is the fact that the S-matrix plays the role of a relative modular invariant of an interacting wedge algebra with respect to that generated by the incoming field. This brings two very different looking properties together: the postulated crossing symmetry of the S-matrix bootstrap approach with the thermal KMS property resulting from wedge

[21] To some of the protagonists of a pure S-matrix theory, the vestiges of QFT were so irritating that they proclaimed a cleansing campaign against it.

localization. In free theories, their relation is pre-empted in the Wigner one-particle theory in which the analytic continuability to wavefunction of antiparticles becomes encoded into the domain properties of the operator $\mathfrak{s}$. For the family of $d = 1 + 1$ factorizing models crossing and KMS are mutually equivalent, whereas in the general setting the problems remain open as the result of incompletely understood domain problems. A manifestation of their close relation in the general case is the uniqueness of the net of local algebras which can be derived from a crossing symmetric S-matrix if one assumes the validity of crossing for generalized form factors. The main motivation arises from the expectation that a constructive approach, which starts with the structure of wedge algebras with their strong on-shell aspects and leaves the issue of point-like fields to the end of the calculations, may reveal something about the true ultraviolet frontier which is presently set by the power-counting in a deformation approach from free fields. Looking at the literature on the S-matrix bootstrap approach of the 1960s, one gets the impression that some authors tried to obtain at least the perturbative on-shell results for the S-matrix without using point-like fields, but they failed in higher orders because unlike local commutativity there was no operational tool for implementing crossing (conjectures as the Mandelstam representation for the elastic scattering problems did not help in this respect). But with the tool of modular theory from AQFT at hand, it may be worthwhile to revisit these old important problems which have not disappeared.

It would be a misunderstanding to think that the new modular-based approach leaves no role for point-like fields. On the contrary, these point-like generators are the carriers of the 'universal modular group' which is an infinite group of unitaries generated by all modular groups of all localization regions with respect to selected reference state vectors. Point-like fields should only be avoided in calculations where they could lead to ambiguities and ultraviolet problems as a consequence of their appearance as 'singular coordinates' concerning short-distance problems. The problem of nontriviality of a theory should not be tied to the ultraviolet aspects of point-like fields, but rather to the nontriviality ($\neq C\mathbf{1}$) of double cone algebras obtained from intersecting wedge algebras. We believe that important physical properties such as the shape of the vacuum polarization clouds generated by applying an operator from such an algebra to the vacuum are determined by the (presently unknown diffuse acting) modular unitary $\Delta^{it}_{\mathcal{O},\Omega}$.

Finally, the present viewpoint of QFT is also very well suited to address a problem which, after lying dormant for a very long time, in recent years has returned to the focus of interest, namely the question whether besides the macrocausal relativistic quantum mechanics mentioned in the introduction and the microcausal local quantum physics with its vacuum polarization structure, there are other relativistic non-microcausal quantum theories [22]. In particular, one would be interested in relativistic theories which permit the physical notion of time-dependent scattering (i.e. obey cluster factorization properties) and which unlike the relativistic mechanics preserve some of the vacuum polarization properties, especially those which are necessary to keep the TCP and spin and statistics theorems and hence the existence of antiparticles as an inexorable consequence of the setting or to find a physically viable new principle from which these extremely important particle physics properties can be obtained in the limit of physics far away from the Planck length. Last but not least, even being extremely lenient on issues of causality and localization, no physical theory which still aims to de-mystify nature can completely sell out on the issue of macrocausality.

All post-renormalization attempts to obtain ultraviolet-improved theories by allowing nonlocal interactions, starting from the Kristensen–Moeller–Bloch [55, 57] replacement of

[22] A recent paper by Lieb and Loss [54] contains an interesting attempt to combine relativistic QM with local quantum field theory. To make this model fully cluster separable (macrocausal) one probably has to combine the localization properties of relativistic quantum mechanics with those of modular localization for the photon field.

point-like Lagrangian interactions by form factors being followed by the Lee–Wick complex pole modification [56] of Feynman rules, up to some of the recent proposals to implement nonlocality via noncommutative spacetime failed on different counts. The old attempts retained Lorentz invariance and unitarity but failed on the starting motivation, namely 'finiteness' [57], not to mention the issue of macrocausality. Of course, even without this motivation of ultraviolet improvement it would be very interesting to know if there are any physically viable nonlocal relativistic theories at all. By this we mean, besides the validity of unitarity and Lorentz invariance, the survival of the physically indispensable macrocausality[23] without which the formalism has no physical interpretation. For the relativistic direct particle interaction theories mentioned in the introduction, this macrocausality was ensured via the cluster separability properties of the S-matrix. Almost 50 years of history on this issue has taught us time and again that the naive idea of a mild modification of point-like Lagrangian interactions which still retains macrocausality under closer scrutiny turned out to be an illusion. In fact the general message is that the notion of a mild violation of microcausality (i.e. maintaining macrocausality) within the standard framework is a questionable concept [59]. One surprising no-go theorem states that if one replaces space-like commutativity by a faster than exponential asymptotic decrease, one falls right back onto local commutativity [60]. This enormous persistence of relativistic localization with sharp borders is corroborated by the use of modular concepts.

These negative results suggest that in order to find a consistent way to get away from local commutativity, one needs a much more radical ansatz which modifies the very spacetime structure. Doplicher, Fredenhagen and Roberts [61] discovered a Bohr–Rosenfeld-like argument which uses a quasiclassical interpretation of the Einstein field equation (coupled with a requirement of absence of measurement-caused black hole 'photon traps') and leads to uncertainty relations of spacetime. Although the initial idea was very conservative, the authors were nevertheless led to quite drastic conceptual changes since any field theoretic localization of observables must be formulated in terms of noncommutative spacetime in which 'points' in the sense of maximal localization correspond to pure states on a quantum mechanical spacetime substrate on which the Poincaré group acts. They found a model which saturates their commutation relations but still maintains the Poincaré symmetry. In more recent times it was realized [62], that when one recasts such models into the setting of Yang–Feldman perturbation theory with a kind of nonlocal interaction, the Lorentz invariance and unitarity of their noncommutative framework can even be upheld in perturbation theory. This is interesting because in many papers [63, 64] which appeared after the DFR work it was claimed that the noncommutative setting leads to inevitable violations of L-invariance and of the optical theorem (unitarity). Most of these incorrect conclusions have their origin in the use of the Feynman formalism without being aware that the iε prescription is no longer the same as the spacetime time-ordering. Interestingly enough, the formalism of Yang–Feldman perturbation theory which works directly with the field equations and seems to be more suitable in this context was precisely the technique used in the first post-renormalization investigations of nonlocal interactions [55].

The main reason for mentioning these noncommutative attempts in the present context is that they keep the Wigner particle picture and the free field Fock space intact but substitute modular localization by something else. The modification consists in passing from the canonical (m, s) Wigner setting to any of the covariant point-like descriptions (83) in section 3 in order to replace the commutative coordinates x_μ in the plane wave factors e^{ipx} by

[23] In the case of form factor modifications of point-like interaction vertices this was shown in [57] and in case of the Feynman rule modifications by complex poles in [58].

operators q_μ fulfilling commutation relations whose structure constants change under Lorentz transformations and in fact form a manifold of skew-symmetric matrices [61] (for fixed structure constants the commutation relations are isomorphic to those of a two-dimensional quantum oscillator. Whereas the modular localized subspaces of the Wigner space can be related via intertwiners to localized test function subspaces, and the modular localization does not depend on the choice of intertwiners (the corresponding point-like fields are in the same local equivalence class), the noncommutative localization leads to an additional spreading[24]. It is an interesting and perhaps even simple problem to decide whether the different canonical versions (belonging to different intertwiners) lead to different noncommutative models or if the concept of local equivalence classes has a noncommutative counterpart.

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Appendix. The abstract spatial modular theory

Suppose that we have a 'standard' spatial modular situation, i.e. a closed real subspace H_R of a complex Hilbert space H such that $H_R \cap \mathrm{i}H_R = \{0\}$ and the complex space $H_D \equiv H_R + \mathrm{i}H_R$ is dense in H. Let e_R and e_I be the projectors onto H_R and $\mathrm{i}H_R$ and define operators

$$t_\pm \equiv \tfrac{1}{2}(e_R \pm e_I). \tag{96}$$

Because of the reality restriction, the two operators have very different conjugation properties: t_+ turns out to be positive $0 < t_+ < \mathbf{1}$, but t_- is antilinear. These properties follow by inspection through the use of the projection and reality properties. There are also some easily derived relations between the projectors and $t_\pm$,

$$e_{R,I}t_+ = t_+(1 - e_{I,R}) \qquad t_+t_- = t_-(1 - t_+) \qquad t_-^2 = t_+(1 - t_+). \tag{97}$$

Theorem 6 ([20]). *In the previous setting, there exist modular objects*[25] J, Δ *and* $S = \mathrm{j}\Delta^{\frac{1}{2}}$ *which reproduce* H_R *as the* $+1$ *eigenvalue real subspace of* S. *They are related to the previous operators by*

$$t_- = J\,|t_-| \qquad \Delta^{\mathrm{i}t} = (1 - t_+)^{\mathrm{i}t}\, t_+^{-\mathrm{i}t}.$$

The proof consists in showing the commutation relation $J\Delta^{\mathrm{i}t} = \Delta^{\mathrm{i}t}J$ ($\curvearrowright J\Delta = \Delta^{-1}J$ since J is anti-unitary) which establishes the dense involutive nature $S^2 \subset 1$ of S by using the previous identities. It is not difficult to show that 0 is not in the point spectrum of $\Delta^{\mathrm{i}t}$.

[24] In the mentioned string-localized cases the string-like spreading is caused by the delocalizing effect of the more complicated intertwiners; it is an intrinsic property of the Wigner representation theory and should be distinguished from the noncommutative spreading.

[25] In the physical application, the Hilbert space can be representation space of the Poincaré group which carries an irreducible positive energy representation or the bigger Fock space of (free or incoming) multiparticle states. In order to have a uniform notation, we use (different from section 2) capital letters for the modular objects and the transformations, i.e. S, J, Δ, $U(a, \Lambda)$.

Corollary 7. *If H_R is standard, then* $\mathrm{i}H_R$, $H_R^{\perp}$ *and* $\mathrm{i}H_R^{\perp}$ *are standard. Here the orthogonality* $\perp$ *refers to the real inner product* $\mathrm{Re}\,(\psi,\varphi)$. *Furthermore, the* J *acts on* H_R *as*

$$JH_R = \mathrm{i}H_R^{\perp}.$$

We leave the simple proofs to the reader (or look up the previous reference [20]). The orthogonality concept is often expressed in the physics literature by $\mathrm{i}H_R^{\perp} = H_R^{\mathrm{symp}\perp}$ referring to symplectic orthogonality in the sense of $\mathrm{Im}(\psi,\varphi)$. There is also a more direct analytic characterization of Δ and J.

Theorem 8 (spatial KMS condition)**.** *The functions* $f(t) = \Delta^{\mathrm{i}t}\psi$, $\psi \in H_R$ *permit an holomorphic continuation* $f(z)$ *holomorphic in the strip* $-\frac{1}{2}\pi < \mathrm{Im}\,z < 0$, *continuous and bounded on the real axis and fulfilling* $f\left(t - \frac{1}{2}\mathrm{i}\right) = Jf(t)$ *which relates the two boundaries. The two commuting operators* $\Delta^{\mathrm{i}t}$ *and j are uniquely determined by these analytic properties, i.e.* H_R *does not admit different modular objects.*

Another important concept in the spatial modular theory is ‘modular inclusion’.

Definition 9 (analogous to Wiesbrock)**.** *An inclusion of a standard real subspace* K_R *into a standard space* $K_R \subset H_R$ *is called ‘modular’ if the modular unitary* $\Delta_{H_R}^{\mathrm{i}t}$ *of* H_R *compresses* K_R *for one sign of t,*

$$\Delta_{H_R}^{\mathrm{i}t} K_R \subset K_R \qquad t < 0.$$

If necessary one adds a $-$ *sign, i.e. if the modular inclusion happens for* $t > 0$ *one calls it a minus modular inclusion.*

Theorem 10. *The modular group of a modular inclusion, i.e.* $\Delta_{K_R}^{\mathrm{i}t}$ *together with* $\Delta_{H_R}^{\mathrm{i}t}$, *generates a unitary representation of the two-parametric affine group of the line.*

The proof consists in observing that the positive operator $\Delta_{K_R} - \Delta_{H_R} \geqslant 0$ is essentially self-adjoint. Hence we can define the unitary group

$$U(a) = \mathrm{e}^{\mathrm{i}\frac{1}{2\pi}a\overline{(\Delta_{K_R}-\Delta_{H_R})}}. \tag{98}$$

the following Borchers relation

$$\Delta_{H_R}^{\mathrm{i}t} U(a) \Delta_{H_R}^{-\mathrm{i}t} = U(\mathrm{e}^{\pm 2\pi t}a) \qquad J_{H_R} U(a) J_{H_R} = U(-a) \tag{99}$$

are consequences. These relations are the dilation-translation commutation relations of the one-dimensional affine group. It would be interesting to generalize this to the modular intersection relation in which case one expects to generate the $SL(2,R)$ group.

The actual situation in physics is the opposite: from group representation theory of certain noncompact groups $\pi(G)$ one obtains candidates for Δ^{it} and J from which one passes to S and H_R. In the case of the Poincaré or conformal group, the boosts or proper conformal transformations in positive energy representations lead to the above situation. The representations do not have to be irreducible; the representation space of a full QFT is also in the application range of the spatial modular theory. If the positive energy representation space is the Fock space over a one-particle Wigner space, the existence of the CCR (Weyl) or CAR functor maps the spatial modular theory into the operator-algebraic modular theory of Tomita and Takesaki. In general such a step is not possible. Connes has given conditions on the spatial theory which lead to the operator-algebraic theory. They involve the facial structure of positive cones associated with the space H_R. Up to now it has not been possible to use them for constructions in QFT. The existing ideas of combining the spatial theory of particles with the Haag–Kastler framework of spacetime localized operator algebras uses the following two facts:

- The wedge algebra $\mathcal{A}(W)$ has known modular objects

$$\Delta^{it} = U(\Lambda_W(-2\pi t)) \qquad J = S_{scat} J_0 \tag{100}$$

whereas the wedge-affiliated L-boost (in fact all $P_+^\uparrow$ transformations) is the same as that of the interacting or free incoming/outgoing theory, the interaction shows up in those reflections which involve time inversion as J. In the latter case, the scattering operator S_{scat} intervenes in the relation between the incoming (interaction-free) J_0 and its Heisenberg counterpart J. In the case of interaction free theories, the J_0 contains in addition to the geometric reflection (basically the TCP) a 'twist' operator which is particularly simple in the case of fermions.
- The wedge algebra $\mathcal{A}(W)$ has PFG generators. In certain cases, these generators have nice (tempered) properties which make them useful in explicit constructions. Two such cases (beyond the standard free fields) are the interacting $d = 1 + 1$ factorizing models and the free anyonic and Wigner spin tower representations; in both cases the PFG property is lost (vacuum polarization is present) for sub-wedge algebras. In the last two Wigner cases, the presence of the twist requires this; only the fermionic twist in the case of $S_{scat} = 1$ is consistent with having PFGs for all localizations.

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